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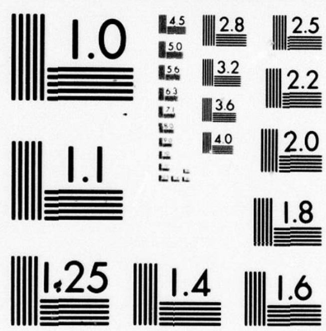

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ON SOME NEW GENERAL AND COMPLEMENTARY ENERGY THEOREMS FOR THE RATE PROBLEMS IN FINITE STRAIN, CLASSICAL ELASTO-PLASTICITY

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Summary: General variational theorems, for the rate problem of classical elasto-plasticity at finite strains, in both Updated Lagrangean (UL) and Total Lagrangean (TL) rate forms, and in terms of alternate measures of stress-rate and conjugate strain-rates, are critically studied from the point of view of their application. Attention is primarily focussed on the derivation of consistent complementary energy rate principles, which could form the basis of consistent and rational assumed stress type finite element methods; and two such principles, in both to UL and TL forms, are newly stated. Systematic procedures to exploit these new principles in the context of a finite element method are also discussed. Also certain general modified variational theorems, to enable an accurate numerical treatment of near incompressible behaviour at large plastic strains, are discussed.

I. Introduction

The advent of high speed digital computers and powerful numerical methods, such as the finite element methods, in the past two decades or so, have greatly expanded the scope of application of nonlinear theories of solid continua to practical problems in engineering. In the formulation of such numerical methods as the finite element methods, for problems of nonlinear solid mechanics, variational theorems (and their generalizations to account for discontinuities at interelement boundaries in a finite element assembly) have played a central role. [See, for instance, WASHIZU (1975), NEMAT-NASSER (1974), ATLURI (1975), and ATLURI and MURAKAWA (1977), for a discussion of finite element formulations in nonlinear elasticity].

Rigorous and consistent formulations for numerical analysis of large strain elasto-plastic problems have become necessary due to the increased interest, in recent years, in analyzing problems such as metal-forming processes, ductile fracture initiation and stable crack growth in cracked bodies, etc. Indeed several such formulations, and applications of the same, have appeared in recent literature. Among these can be cited the works of: HIBBIT, MARCAL, and RICE (1970), who use a total Lagrangean (TL) rate formulation [wherein a fixed reference frame is used]; NEEDLEMAN (1972), NEEDLEMAN and TVERGAARD (1977), and HUTCHINSON (1973) who also use a TL rate formulation, but with convected coordinates; YAMADA, HIRAKAWA, and WIFI (1977) who use an Updated Lagrangean (UL) rate formulation [wherein the current configuration is used as a reference for the subsequent step]; OSIAS (1972) who also uses an UL scheme, which, due to the use of an elastic-plastic rate constitutive law that does not admit to a potential, leads to non-symmetric stiffness matrices through a Galerkin Scheme; McMEEKING and RICE (1975) who also use a UL scheme which, through the use of a rate constitutive law with a potential, leads to symmetric stiffnesses; and NEMAT-NASSER and TAYA (1976) whose formulation represents a modification to that of McMEEKING and RICE (1975) to improve the accuracy in the case of large deformation of compressible materials. All the above cited works employ a classical rate-independent elasto-plastic theory, as generalized by HILL (1959). It should also be noted that all the above finite element rate formulations are based on the principle of virtual work in rate form, as first stated by

HILL (1959); and thus all the above finite element schemes are based on assumed displacements that are compatible at interelement boundaries.

However, to the best of the author's knowledge, no studies concerning the convergence of the assumed displacement finite element methods of the above cited type, for the rate problems of classical elasto-plasticity, exist in the literature. Even in the somewhat simpler problem of finite elasticity, studies of convergence of finite element methods, based on potential energy principles, are just beginning to emerge [ODEN (1978)]. From this standpoint, as well as from that of possibly studying solution bounds, it is of interest to consider consistent formulations of numerical (finite element) methods based on complementary energy principles for the rate problem of finite strain elasto-plasticity.

Another important question in numerical schemes for elastic-plastic flow at large strains is how to deal with the effectively incompressible behaviour at such magnitudes of strain. It is well-known that numerical schemes based directly on the principle of virtual work fail in the limit of incompressibility unless the mean stress is introduced as an additional variable in the formulation. Such formulations, which are essentially variations of the well-known HELLINGER (1914) - REISSNER (1950) theorem, were introduced for nearly or precisely incompressible linear elastic materials by HERRMANN (1965), TAYLOR, PISTER, and HERRMANN, (1968), and by KEY (1969). To improve the numerical accuracy in the near-incompressible case, NAGTEGAAL, PARKS, and RICE (1974, Appendix 2) modify their UL rate potential energy formulation for elasto-plasticity, in a way analogous to that of KEY (1969), except, instead of the mean pressure as in KEY (1969), they use the dilatational strain rate as an independent variable. As a consequence, even though the formulation of KEY is valid for both nearly and precisely incompressible cases, the formulation of NAGTEGAAL et.al ceases to be valid in the case of precise incompressibility. However, due to the inherent nature of the complementary energy principle (with assumed stresses as variables), it is much easier to treat situations of near or precise incompressiblity when finite element schemes based on a complementary energy principle are used. The works of TONG (1969), and PIAN and LEE (1976) in solving problems of linear elastic near-incompressible solids; and that of MURAKAWA and ATLURI (1978b) in solving finite strain problems of incompressible nonlinear elastic solids, all of which are based on appropriate complementary energy principles, tend to support the above view. Moreover, as is well-known, better solutions for stresses are obtained from numerical schemes based on complementary energy principles than from those based on potential energy principles (wherein, the stress solution is obtained by differentiation of the numerical solution for displacements, which results in a loss of accuracy for stresses).

Also, as noted by MASUR and POPELAR (1976), the complementary energy approach holds a considerable promise for applications to buckling problems, wherein, an approximation to the stress state before buckling is often possible even when displacements remain unknown.

For the above reasons, special emphasis is placed in the present paper on the study of the existence of consistent complementary energy rate theorems for the rate problem of finite strain classical, rate-independent, elasto-plasticity. Both the types of formulations, viz., the Total Lagrangean as well as the Updated Lagrangean, are considered. In this process two new consistent rate complementary energy principles for the rate problem of finite strain classical elasto-plasticity have been found. Systematic procedures to exploit these complementary rate principles in the context of assumed stress finite element methods are also discussed. In addition, a critical evaluations of the relative effectiveness of general rate principles, in both TL and UL rate forms, and in terms of alternate stress-rate and conjugate strain-rate measures, for application to numerical analysis of finite strain elasto-plasticity problems, is made. Also included in the present paper are certain general modified variational theorems which are of significance in the numerical treatment of near incompressible behaviour at large magnitudes of plastic strain.

II. Preliminaries:

For simplicity we refer all configurations of the body to a fixed rectangular cartesian coordinate system. We adopt the notation: (-) under symbol denotes a vector; (~) under symbol denotes a second-order tensor; $\underline{a} = \underset{\sim}{A} \cdot \underline{b}$ implies that $a_i = A_{ik}\, b_k$; $\underset{\sim}{A} \cdot \underset{\sim}{B}$ denotes product of two tensors such that $(\underset{\sim}{A} \cdot \underset{\sim}{B})_{ij} = A_{ik}\, B_{kj}$; $(\underset{\sim}{A} : \underset{\sim}{B}) = \text{trace}\ (\underset{\sim}{A}^T \cdot \underset{\sim}{B}) = A_{ij}B_{ij}$; and $\underline{u} \cdot \underline{t} = u_i t_i$.

A particle in the undeformed body has a position vector $\underline{x} = (x_\alpha \underline{e}_\alpha)$ where $\underline{e}_\alpha$ are unit cartesian bases. We adopt the notation $\underline{\nabla}^o = (\underline{e}_\alpha\, \partial/\partial x_\alpha)$, in the undeformed configuration, C_o. The position vector of the particle in the current deformed state, say C_N, is $\underline{y} = (y_i \underline{e}_i)$. We also use the notation that $\underline{\nabla}^N = (\underline{e}_i\, \partial/\partial y_i)$. The gradient of $\underline{y}$ is the tensor $\underset{\sim}{F}$, ie., $\underset{\sim}{F} = (\underline{\nabla}^o \underline{y})^T$; $F_{i,\alpha} = y_{i,\alpha} \equiv \partial y_i/\partial x_\alpha$. We also note that the base vectors of the convected coordinates x_α in the current deformed configuration, C_N, are given by, $\underline{g}_\alpha = \underline{e}_i\, y_{i,\alpha}$. The non-singular tensor $\underset{\sim}{F}$ is considered to have the polar-decomposition, $\underset{\sim}{F} = \underset{\sim}{\alpha} \cdot (\underset{\sim}{I} + \underset{\sim}{h})$, where $(\underset{\sim}{I} + \underset{\sim}{h})$ is a symmetric positive definite tensor called the stretch tensor, $\underset{\sim}{I}$ is an identity tensor, and $\underset{\sim}{\alpha}$ is an orthogonal rotation tensor such that $\underset{\sim}{\alpha}^T = \underset{\sim}{\alpha}^{-1}$. The deformation tensor $\underset{\sim}{G}$ is defined by $\underset{\sim}{G} = \underset{\sim}{F}^T \cdot \underset{\sim}{F} \equiv (\underset{\sim}{I} + \underset{\sim}{h})^2$. The Green-Lagrange strain tensor is defined by $\underset{\sim}{g} = 1/2\ (\underset{\sim}{G} - \underset{\sim}{I}) = 1/2\ (\underset{\sim}{e} + \underset{\sim}{e}^T + \underset{\sim}{e}^T \cdot \underset{\sim}{e})$ where $\underset{\sim}{e}$ is the gradient of the displacement vector $\underline{u}$ ($\equiv \underline{y} - \underline{x}$), ie., $\underset{\sim}{e} = (\underline{\nabla}^o \underline{u})^T$ such that $e_{i\alpha} = u_{i,\alpha}$.

For our present purposes, we introduce the stress measures: (i) the true (Cauchy) stress tensor $\underset{\sim}{\tau}$ ($\equiv \tau_{ij}\,\underline{e}_i\underline{e}_j \equiv \tau^{\alpha\beta}\underline{g}_\alpha\underline{g}_\beta$); (ii) a weighted stress tensor, generally referred to as the Kirchhoff stress tensor, $\underset{\sim}{\sigma} = J\underset{\sim}{\tau}$ ($\equiv \sigma^{\alpha\beta}\underline{g}_\alpha\underline{g}_\beta \equiv J\tau^{\alpha\beta}\underline{g}_\alpha\underline{g}_\beta$, where J is the determinent of the matrix $[y_{i,\alpha}]$); (iii) the Piola-Lagrange or the First Piola-Kirchhoff stress tensor, $\underset{\sim}{t} \equiv$ ($t_{\alpha j}\,\underline{e}_\alpha\underline{e}_j$); and (iv) the second Piola-Kirchhoff stress tensor, $\underset{\sim}{s}$ ($\equiv s_{\alpha\beta}\,\underline{e}_\alpha\underline{e}_\beta$). As discussed, for instance, by TRUESDELL and NOLL (1965) and FRAEIJS DE VEUBEKE (1972), the relations between the above stress measures are seen to be:

$$\underset{\sim}{\tau} = \frac{1}{J}\underset{\sim}{F}\cdot\underset{\sim}{t} = \frac{1}{J}\underset{\sim}{F}\cdot\underset{\sim}{s}\cdot\underset{\sim}{F}^T \tag{1}$$

$$\underset{\sim}{t} = \underset{\sim}{s}\cdot\underset{\sim}{F}^T = J(\underset{\sim}{F}^{-1}\cdot\underset{\sim}{\tau}) \tag{2}$$

and

$$\underset{\sim}{s} = J(\underset{\sim}{F}^{-1}\cdot\underset{\sim}{\tau}\cdot\underset{\sim}{F}^{-T}) = \underset{\sim}{t}\cdot\underset{\sim}{F}^{-T} \tag{3}$$

where $\underset{\sim}{F}^{-T} \equiv (\underset{\sim}{F}^{-1})^T$. We also note that eventhough $\underset{\sim}{\sigma}$ and $\underset{\sim}{s}$ are distinctly different tensors, they have the interesting property that $\sigma^{\alpha\beta} \equiv s_{\alpha\beta}$ where the contravariant components $\sigma^{\alpha\beta}$ and the components $s_{\alpha\beta}$ are as defined earlier. Finally we introduce a stress measure which is labelled by FRAEIJS DE VEUBEKE (1972) as the Jaumann Stress, $\underset{\sim}{r}$, through the relation,

$$\underset{\sim}{r} = 1/2\,(\underset{\sim}{t}\cdot\underset{\sim}{\alpha} + \underset{\sim}{\alpha}^T\cdot\underset{\sim}{t}^T) \tag{4}$$

$$= 1/2\left[\underset{\sim}{s}\cdot(\underset{\sim}{I}+\underset{\sim}{\bar{h}}) + (\underset{\sim}{I}+\underset{\sim}{\bar{h}})\cdot\underset{\sim}{s}\right] \tag{5}$$

It is seen from the above that the stress tensors $\underset{\sim}{\tau}$, $\underset{\sim}{\sigma}$, $\underset{\sim}{s}$, and $\underset{\sim}{r}$ are symmetric, while $\underset{\sim}{t}$ is unsymmetric.

Also, as shown, for instance, by TRUESDELL and NOLL (1965) and FRAEIJS DE VEUBEKE (1972), the linear momentum balance (LMB) equation, the angular momentum balance (AMB) equation, the traction boundary condition (TBC), and the displacement boundary condition (DBC) can be written as follows:

LMB:

$$\underline{\nabla}^N \cdot \underset{\sim}{\tau} + \rho^N \underline{B} = o \tag{6}$$

or

$$\underline{\nabla}^o \cdot (\underset{\sim}{s} \cdot \underset{\sim}{F}^T) + \rho^o \underline{B} = o \tag{7}$$

or

$$\underline{\nabla}^o \cdot \underset{\sim}{t} + \rho^o \underline{B} = o \tag{8}$$

where $\underline{B}$ is the body force per unit mass, and ρ^N and ρ^o are the mass densities per unit volume in C_N and C_o, respectively.

AMB:

$$\underset{\sim}{\tau} = \underset{\sim}{\tau}^T \quad (9) \qquad \text{or} \qquad \underset{\sim}{s} = \underset{\sim}{s}^T \tag{10}$$

or

$$\underset{\sim}{F} \cdot \underset{\sim}{t} = \underset{\sim}{t}^T \cdot \underset{\sim}{F}^T \qquad \text{or} \qquad (\underset{\sim}{h} + \underset{\sim}{I}) \cdot \underset{\sim}{t} \cdot \underset{\sim}{\alpha} = \text{symmetric} \tag{11a,b}$$

TBC:

$$\underline{\bar{t}} = \underline{n} \cdot \underset{\sim}{t} = \underline{n} \cdot (\underset{\sim}{s} \cdot \underset{\sim}{F}^T) \tag{12a,b}$$

where $\underline{n}$ is a unit normal to the surface S_{σ_o}, in configuration C_o, where tractions are prescribed to be $\underline{\bar{t}}$ per unit area.

DBC:

$$\underline{u} = \underline{\bar{u}} \text{ on } S_{u_o} \tag{12c}$$

where S_{u_o} is the surface in C_o where displacements are prescribed to be $\underline{\bar{u}}$. It is possible to have a mixed-mixed problem wherein, at any point on the surface of the body in C_o certain components of tractions and the conjugate components of displacements may be simultaneously prescribed.

In connection with the rate formulations of classical elasto-plasticity, the requirements for a suitable stress rate are now well recognized as: that the stress rate vanishes when the solid continuum undergoes a rigid motion alone and when the stress tensor is referred to a coordinate system undergoing the same motion; and that the rate of invariants of the stress tensor is stationary when the stress rate vanishes. The questions of objective stress rates and their use in classical rate theories of plasticity have been discussed by several authors; for instance, OLDROYD (1950, 1958), TRUESDELL (1955), COTTER and RIVLIN (1955), PRAGER (1961), SEDOV (1966), MASUR (1961), NAGHDI and WAINWRIGHT (1961), and HILL (1967). With this background, we now discuss the following rate variational principles in the rate theory of classical elasto-plasticity.

III. Finite Strain Elastic-Plastic Analysis:

3.1 Rate Variational Principles in Updated Lagrangean (UL) Formulation

In the UL formulation, we refer the solution variables (displacements, strains, and stresses) in the state C_{N+1} to the configuration of the body in the immediately preceding state, C_N, which is pressumed to be known. Let y_i^N be the current (in state C_N) Cartesian spatial coordinates of a particle, to be used as a reference system for the current increment, ie., from C_N to C_{N+1}. Let $\underset{\sim}{\tau}^N$ be the true stress in C_N. Thus in the UL rate formulation, one is concerned essentially with an initial stress problem, whereas the initial displacements in C_N as referred to C_N itself are clearly zero. Let $\underset{\sim}{\dot{t}}$, $\underset{\sim}{\dot{s}}$, and $\underset{\sim}{\dot{\tau}}$ represent the rates of the Piola-Lagrange stress, 2^{nd} Piola-Kirchhoff stress, and Jaumann stress, respectively, referred to the current configuration. Further, we note that $\underset{\sim}{\dot{s}} = \underset{\sim}{s}_N^{N+1} - \underset{\sim}{\tau}^N$; $\underset{\sim}{\dot{t}} = \underset{\sim}{t}_N^{N+1} - \underset{\sim}{\tau}^N$, etc., where $\underset{\sim}{s}_N^{N+1}$ is defined as the 2^{nd} Piola-Kirchhoff stress in state C_{N+1} as referred to the configuration C_N (ie., measured per unit area in C_N), etc.,. Let $\underline{\nabla}^N$ represent the gradient operator in the current coordinates, and $\underline{\dot{u}}$ be the rate of deformation from the current configuration. We define the rate of displacement gradient $\underset{\sim}{\dot{e}} \equiv (\underline{\nabla}^N \underline{\dot{u}})^T$ and write $\underset{\sim}{\dot{e}} = \underset{\sim}{\dot{\epsilon}} + \underset{\sim}{\dot{\omega}}$, where $\underset{\sim}{\dot{\epsilon}} \left[\dot{\epsilon}_{ij} = 1/2\ (\partial \dot{u}_i/\partial y_j^N + \partial \dot{u}_j/\partial y_i^N) \right]$

is the UL Strain rate and $\underset{\sim}{\omega}\left[\dot{\omega}_{ij} = 1/2\ (\partial \dot{u}_i/\partial y_j^N - \partial \dot{u}_j/\partial y_i^N)\right]$ is the spin-rate.

3:1:0: Rate Potentials:

We also use $\overset{*}{\underset{\sim}{\dot{\sigma}}}$ to denote the corotational rate (or what is also usually called the "Jaumann rate") of Kirchhoff stress, $\underset{\sim}{\sigma}$. Based on general discussions concerning stress rates, contained, for instance, in the References cited earlier, it is seen that;

$$\underset{\sim}{\dot{s}} = \underset{\sim}{\dot{\sigma}}^* - \underset{\sim}{\dot{\epsilon}} \cdot \underset{\sim}{\tau}^N - \underset{\sim}{\tau}^N \cdot \underset{\sim}{\dot{\epsilon}} \tag{13}$$

and

$$\underset{\sim}{\dot{t}} = \underset{\sim}{\dot{\sigma}}^* - \underset{\sim}{\dot{\epsilon}} \cdot \underset{\sim}{\tau}^N - \underset{\sim}{\tau}^N \cdot \underset{\sim}{\omega} \tag{14}$$

Considering a classical elasto-plastic theory, it has been noted by HILL (1967a,b) that a rate potential, $\dot{V}$, can be written for $\underset{\sim}{\dot{\sigma}}^*$ such that,

$$\underset{\sim}{\dot{\sigma}}^* = \partial \dot{V}/\partial \underset{\sim}{\dot{\varepsilon}} \tag{15}$$

As also noted by HILL (1967a), the form of the rate potential $\dot{V}$ can be written as:

$$\dot{V} = 1/2\ L_{ijk\ell}\dot{\epsilon}_{ij}\dot{\epsilon}_{k\ell} - \frac{\alpha}{g}\ (\lambda_{k\ell}\dot{\epsilon}_{k\ell})^2 \tag{16}$$

which yields a bi-linear constitutive law through Eq. (15). Following HILL (1967a) we note that, in Eq. (16), $L_{ijk\ell}$ is a tensor of instantaneous elastic modulii, assumed to be positive definite and symmetric under $ij \longleftrightarrow k\ell$ interchange, $\alpha = 1$, or 0 according as $\lambda_{k\ell}\dot{\epsilon}_{k\ell}$ is positive or negative; λ_{ij} is a tensor normal to the hyperplane interface between elastic and plastic domain in the strain-rate space; while g is a scalar related to the measure of rate of hardening due to plastic deformation. Prandtl-Reuss type rate equations of the form of Eq. (15) for classical isotropically hardening materials have been used by several authors, for instance, HUTCHINSON (1973), McMEEKING and RICE (1975), and NEMAT-NASSER and TAYA (1976).

From Eqs. (13) and (15) it is seen that if a rate potential $\dot{V}$ exists for $\underset{\sim}{\sigma}^*$, then a potential $\dot{W}$ also exists for $\dot{\underset{\sim}{s}}$, such that,

$$\dot{\underset{\sim}{s}} = \partial\dot{W}/\partial\dot{\underset{\sim}{\varepsilon}} \tag{17}$$

and further it is seen that

$$\dot{W} = \dot{V} - \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\varepsilon}} \cdot \dot{\underset{\sim}{\varepsilon}}) \tag{18}$$

Likewise, from Eqs. (14, 15 and 13), it can also be seen that a rate potential $\dot{U}$ for $\dot{\underset{\sim}{t}}$ also exists such that,

$$\dot{\underset{\sim}{t}} = \partial\dot{U}/\dot{\underset{\sim}{e}}^T \tag{19}$$

where

$$\dot{U} = \dot{V} - \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\varepsilon}} \cdot \dot{\underset{\sim}{\varepsilon}}) + 1/2\, \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{e}}^T \cdot \dot{\underset{\sim}{e}}) \tag{20}$$

$$\equiv \dot{W} + 1/2\, \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{e}}^T \cdot \dot{\underset{\sim}{e}}) \tag{21}$$

Further, by applying the polar-decomposition theorem, we see that,

$$\underset{\sim}{F}_N^{N+1} \equiv (\underline{\nabla}^N \underline{y}^{N+1})^T = \underset{\sim}{\alpha}_N^{N+1} \cdot (\underset{\sim}{I} + \dot{\underset{\sim}{h}}) \tag{22}$$

Where $\underset{\sim}{F}_N^{N+1}$ and $\underset{\sim}{\alpha}_N^{N+1}$ are the deformation gradient and rotation tensors, respectively, in C_{N+1} as referred to C_N; and $\dot{\underset{\sim}{h}}$ is the UL rate of stretch. Writing $\underset{\sim}{\alpha}_N^{N+1} = \underset{\sim}{I} + \dot{\underset{\sim}{\alpha}}$ (where $\dot{\underset{\sim}{\alpha}}$ is the UL rate of rotation), it is seen from Eq. (22) that,

$$\dot{\underset{\sim}{e}} = \dot{\underset{\sim}{\alpha}} + \dot{\underset{\sim}{h}} \tag{23}$$

Thus, as may be expected, it is seen that in the UL rate formulation, $\dot{\underset{\sim}{h}} \equiv \dot{\underset{\sim}{\varepsilon}}$ and $\dot{\underset{\sim}{\alpha}} \equiv \dot{\underset{\sim}{\omega}}$ where $\dot{\underset{\sim}{\varepsilon}}$ and $\dot{\underset{\sim}{\omega}}$ are as defined earlier. From the definition of the Jaumann stress $\underset{\sim}{r}$, as given in Eqs. (4,5), the UL rate $\dot{\underset{\sim}{r}}$ is seen to be given by;

$$\dot{\underset{\sim}{r}} = 1/2\ [\dot{\underset{\sim}{t}} + \dot{\underset{\sim}{t}}^T + \underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{\alpha}} + \dot{\underset{\sim}{\alpha}}^T \cdot \underset{\sim}{\tau}^N] \tag{24a}$$

$$\equiv 1/2\ [\dot{\underset{\sim}{t}} + \dot{\underset{\sim}{t}}^T + \underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{\omega}} + \dot{\underset{\sim}{\omega}}^T \cdot \underset{\sim}{\tau}^N] \tag{24b}$$

or, also, $$\dot{\underset{\sim}{r}} = \dot{\underset{\sim}{s}} + 1/2\ (\underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{\epsilon}} + \dot{\underset{\sim}{\epsilon}} \cdot \underset{\sim}{\tau}^N) \tag{25}$$

comparing Eqs. (13) and (25), it is seen that,

$$\dot{\underset{\sim}{r}} = \dot{\underset{\sim}{\sigma}}^* - 1/2(\dot{\underset{\sim}{\epsilon}} \cdot \underset{\sim}{\tau}^N + \underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{\epsilon}}) \tag{26}$$

Thus if $\dot{V}$ is a rate potential for $\dot{\underset{\sim}{\sigma}}^*$, it then follows from Eq. (26) that there exists a rate potential $\dot{Q}$ for $\dot{\underset{\sim}{r}}$, such that

$$\dot{\underset{\sim}{r}} = \partial\dot{Q}/\partial\dot{\underset{\sim}{h}} \equiv \partial\dot{Q}/\partial\dot{\underset{\sim}{\epsilon}} \tag{27}$$

where, $$\dot{Q} = \dot{V} - 1/2\ \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\epsilon}} \cdot \dot{\underset{\sim}{\epsilon}}) \tag{28}$$

Results given in Eqs. (27) and (28) are useful, as shown later on, in formulating consistent complementary energy rate principles for the rate theory of finite strain plasticity.

With the above considerations for rate potentials for alternate stress-rates, we proceed to formulate the boundary value problem in terms of the piecewise linear incremental field equations and boundary conditions in the UL rate form, as follows:

3.1.1: Field Equations and Boundary conditions in UL Rate Form:

In terms of $\dot{\underset{\sim}{s}}$, $\dot{\underset{\sim}{\epsilon}}$ and $\dot{\underline{u}}$

Considering, for instance, the linear momentum balance equation for $\underset{\sim}{s}_N^{N+1}$ in the UL Coordinates, we see from Eq. (7) that this can be written as:

$$\underline{\nabla}^N \cdot \left[\underset{\sim}{s}_N^{N+1} \cdot (\underset{\sim}{F}_N^{N+1})^T\right] + \rho^N\ \underline{B}^{N+1} = 0 \tag{29}$$

where, as defined earlier, $\underset{\sim}{s}_N^{N+1}$ is the 2nd Piola-Kirchhoff stress in C_{N+1} as measured per unit area in C_N and $\underset{\sim}{s}_N^{N+1} = \underset{\sim}{\tau}^N + \dot{\underset{\sim}{s}}$ etc. The initial stress field $\underset{\sim}{\tau}^N$, which is assumed to be equilibrated (in an actual numerical implementation this may not be true, and hence it is often necessary to employ corrective iteration procedures to check the true equilibrium in C_N), is then required to satisfy the equation,

$$\underline{\nabla}^N \cdot \underset{\sim}{\tau}^N + \rho^N \underline{B}^N = o \tag{30}$$

Comparing Eqs. (29) and (30) we obtain the rate form of the linear momentum balance equation for $\dot{\underset{\sim}{s}}$. Using arguments similar to the above, the following field equations and boundary conditions are derived in UL rate form.

(LMB)→ $$\underline{\nabla}^N \cdot [\dot{\underset{\sim}{s}} + \underset{\sim}{\tau}^N \cdot (\underline{\nabla}^N \dot{\underline{u}})] + \rho^N \dot{\underline{B}} = 0 \tag{31}$$

(AMB)→ $$\dot{\underset{\sim}{s}} = \dot{\underset{\sim}{s}}^T \tag{32}$$

(Compatibility)→ $$\dot{\underset{\sim}{g}} \equiv \dot{\underset{\sim}{\varepsilon}} = 1/2(\dot{\underset{\sim}{e}} + \dot{\underset{\sim}{e}}^T) \equiv 1/2\left[(\underline{\nabla}^N \dot{\underline{u}}) + (\underline{\nabla}^N \dot{\underline{u}})^T\right] \tag{33}$$

(TBC)→ $$\underline{n}^* \cdot [\dot{\underset{\sim}{s}} + \underset{\sim}{\tau}^N \cdot (\underline{\nabla}^N \dot{\underline{u}})] \equiv \dot{\underline{t}} = \dot{\bar{\underline{t}}} \text{ at } S_{\sigma_N} \tag{34}$$

(DBC)→ $$\dot{\underline{u}} = \dot{\bar{\underline{u}}} \text{ at } S_{u_N} \tag{35}$$

where ρ^N is the mass density in C_N; $\dot{\underline{B}}$ are rate of body forces/unit mass; S_{σ_N}, and S_{u_N} are appropriate segments of the boundary of the solid in C_N; and $\underline{n}^*$ is a unit normal to the boundary of the solid in C_N.

<u>In terms of $\dot{\underset{\sim}{t}}$, $\dot{\underset{\sim}{e}}$, and $\dot{\underline{u}}$</u>

In manner analogous to above, these field equations can be shown to be:

(LMB)→ $$\underline{\nabla}^N \cdot \dot{\underset{\sim}{t}} + \rho^N \dot{\underline{B}} = o \tag{36}$$

(AMB)→ $$(\underline{\nabla}^N \dot{\underline{u}})^T \cdot \underset{\sim}{\tau}^N + \dot{\underset{\sim}{t}} = \dot{\underset{\sim}{t}}^T + \underset{\sim}{\tau}^N \cdot (\underline{\nabla}^N \dot{\underline{u}}) \tag{37}$$

or, equivalently, (AMB) can also be written as:

(AMB)→

$$\dot{\underset{\sim}{\alpha}} \cdot \underset{\sim}{\tau}^N + \dot{\underset{\sim}{h}} \cdot \underset{\sim}{\tau}^N + \dot{\underset{\sim}{t}} = \dot{\underset{\sim}{t}}^T + \underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{h}} + \underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{\alpha}}^T \tag{38}$$

or $$\dot{\underset{\sim}{h}} \cdot \underset{\sim}{\tau}^N + \underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{\alpha}} + \dot{\underset{\sim}{t}} = \dot{\underset{\sim}{t}}^T + \underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{h}} + \dot{\underset{\sim}{\alpha}}^T \cdot \underset{\sim}{\tau}^N \tag{38a}$$

(Compatibility)→ $$\dot{\underset{\sim}{e}} = (\underline{\nabla}^N \dot{\underline{u}})^T \tag{39}$$

or, equivalently, $$(\underline{\nabla}^N \dot{\underline{u}})^T = \dot{\underset{\sim}{\alpha}} + \dot{\underset{\sim}{h}} \tag{40}$$

(TBC)→ $$\underline{n}^* \cdot \dot{\underset{\sim}{t}} = \dot{\bar{\underline{t}}} \text{ at } S_{\sigma_N} \tag{41}$$

(DBC)→ same as in Eq. (35).

3.1.2 General Variational Principles in UL Rate Form:

3.1.2.1: In terms of $\dot{\underset{\sim}{s}}$, $\dot{\underset{\sim}{\epsilon}}$ and $\dot{\underline{u}}$:

Using a virtual work principle as applicable to an initial stress problem, and following the procedure outlined by WASHIZU (1975), we obtain a general UL rate principle for elastic-plastic problems, analogous to the well known HU (1955) - WASHIZU (1955) principle of linear elasticity. This general rate principle* governing Eqs. (31 to 35 and 17) can be stated as the condition of stationarity of the functional:

$$\begin{aligned} \pi^{*2}_{HW}(\dot{\underline{u}},\dot{\underset{\sim}{\epsilon}},\dot{\underset{\sim}{s}}) = \int_{V_N} \Big\{ & \dot{W}(\dot{\underset{\sim}{\epsilon}}) - \rho^N \dot{\underline{B}} \cdot \dot{\underline{u}} + 1/2\, \underset{\sim}{\tau}^N : [(\underline{\nabla}^N \dot{\underline{u}}) \cdot (\underline{\nabla}^N \dot{\underline{u}})^T] \\ & - \dot{\underset{\sim}{s}} : [\dot{\underset{\sim}{\epsilon}} - 1/2 < (\underline{\nabla}^N \dot{\underline{u}}) + (\underline{\nabla}^N \dot{\underline{u}})^T >] \Big\} dv - \int_{S_{\sigma_N}} \dot{\bar{\underline{t}}} \cdot \dot{\underline{u}}\, ds \\ & - \int_{S_{u_N}} \dot{\underline{t}} \cdot (\dot{\underline{u}} - \dot{\bar{\underline{u}}}) ds. \end{aligned} \tag{42}$$

where $\dot{W}$ is a rate potential for $\dot{\underset{\sim}{s}}$, as defined through Eqs. (18 and 16). The above rate variational principle governs the rate variables from C_N to C_{N+1}.

*The general variational principles as stated in Eq. (42) above, as well as those in Eqs. (53, 69, 82, 85, 114, 119, and 127) to follow, can be modified appropriately through the method of Lagrange Multipliers, as discussed in ATLURI and MURAKAWA (1977), to account for discontinuities at interelement boundaries when these principles are applied to a finite element assembly.

The satisfaction of the fully nonlinear field equations in C_N, in a numerical solution method such as the finite element method, must be checked at each step; and these checks can be performed based on a variational principle governing the nonlinear field equations at C_N. The details of such corrective iterative procedures, at the end of each increment, as the "equilibrium check", "compatibility mismatch check", etc. can be found, for instance, in the thesis by MURAKAWA (1978).

We now consider certain special cases of the general rate principle given through Eq. (42). If Eqs. (17, 33, and 35) are met a priori, one can reduce Eq. (42) to a functional associated with the UL rate principle of potential energy, as:

$$\pi_p^{*2}(\dot{\underline{u}}) = \int_{V_N} \left\{\dot{W}(\dot{\underline{u}}) - \rho^N \dot{\underline{B}} \cdot \dot{\underline{u}} + 1/2\, \underset{\sim}{\tau}^N : \left[(\underline{\nabla}^N \dot{\underline{u}}) \cdot (\underline{\nabla}^N \dot{\underline{u}})^T\right]\right\} dv - \int_{S_{\sigma_N}} \dot{\bar{\underline{t}}} \cdot \dot{\underline{u}}\, ds \tag{43}$$

The principle $\delta\pi_p^{*2} = 0$ can be seen to lead to Eqs. (31,32, and 34), and is equivalent to a principle stated originally by HILL (1959).

By inverting the relation in Eq. (17) to express $\dot{\underset{\sim}{\epsilon}}$ interms of $\dot{\underset{\sim}{s}}$, one can achieve a contact transformation,

$$\dot{W} - \dot{\underset{\sim}{s}} : \dot{\underset{\sim}{\epsilon}} = -\dot{S}^*(\dot{\underset{\sim}{s}}) \tag{44}$$

Using Eq. (44) one can eliminate $\dot{\underset{\sim}{\epsilon}}$ from Eq. (42) to derive a HELLINGER (1914) - REISSNER (1950) type UL rate principle, with the associated functional:

$$\pi_{HR}^{*2}(\dot{\underline{u}}, \dot{\underset{\sim}{s}}) = \int_{V_N} \left\{- \dot{\underset{\sim}{S}}^*(\dot{\underset{\sim}{s}}) - \rho^N \dot{\underline{B}} \cdot \dot{\underline{u}} + 1/2\, \underset{\sim}{\tau}^N : \left[(\underline{\nabla}^N \dot{\underline{u}}) \cdot (\underline{\nabla}^N \dot{\underline{u}})^T\right] + 1/2\, \dot{\underset{\sim}{s}} : \left[(\underline{\nabla}^N \dot{\underline{u}}) + (\underline{\nabla}^N \dot{\underline{u}})^T\right]\right\} dv - \int_{S_{\sigma_N}} \dot{\bar{\underline{t}}} \cdot \dot{\underline{u}}\, ds - \int_{S_{u_N}} \dot{\underline{t}} \cdot (\dot{\underline{u}} - \dot{\bar{\underline{u}}}) ds \tag{45}$$

Based on the arguments presented earlier for a linear elastic case, for instance, by ATLURI (1975), it can be seen that in a finite element application of the principle stated through Eq. (45), one needs to assume over each finite element, an arbitrary, symmetric, and differentiable stress-rate field $\dot{\tilde{s}}$, and a differentiable $\dot{\underline{u}}$ that is also inherently compatible at the interelement boundaries.

We now examine the possibility of deriving a complementary energy rate principle from Eq. (45). To this end, we first note that in the present UL rate formulation the LMB conditions Eq. (31), are linear in $\dot{\tilde{s}}$ and further, unlike in Eq. (7), do not involve coupling of $\dot{\tilde{s}}$ with displacement gradients. Thus it becomes possible to satisfy both the LMB and AMB conditions, Eqs. (31 and 32) respectively, a priori, by choosing a symmetric $\dot{\tilde{s}}$ such that,

$$\dot{\tilde{s}} = \text{curl curl } \tilde{A} + \dot{\tilde{s}}^P \tag{46}$$

where $\tilde{A}$ is the symmetric Maxwell-Morera-Beltrami second-order stress function tensor for a general three-dimensional case. In Eq. (46) curl $\tilde{A}$ is defined such that $(\text{curl } \tilde{A})_{ij} = e_{ipk} A_{jk,p}$; $(\text{curl curl } \tilde{A})_{ij} = e_{imn} e_{jpq} A_{mp,nq}$; e_{ijk} is the alternating tensor, and $\dot{\tilde{s}}^P$ is any symmetric particular solution such that,

$$\underline{\nabla}^N \cdot \dot{\tilde{s}}^P = -\rho^N \dot{\underline{B}} - \underline{\nabla}^N \cdot [\underline{\tau}^N \cdot (\underline{\nabla}^N \dot{\underline{u}})] \tag{47}$$

one simple way of satisfying Eq. (47) is to assume particular solutions for the direct stresses $\dot{s}_{ii}$ (no sum on i; i = 1,2,3) only, in the following way.

$$\dot{s}^P_{ii} = \int_{y^N_i} [-\rho^N \dot{B}_i - (\tau^N_{kj} \dot{u}_{i;j})_{;k}] dy^N_i \tag{48}$$

(no sum on i, i = 1,2,3)

and $\qquad \dot{s}^P_{ij} = 0 \qquad (i \neq j) \qquad\qquad (49)$

In Eq. (48) (;i) indicates $\partial(\ )/\partial y_i^N$. However, if the above assumptions are used in an assumed-stress type numerical scheme (discussed further below) the question of completeness of the chosen stresses, or; in other words, the effect, on that numerical solution, of the lack of account of the influence of displacement rates on the chosen shear-stress-rate field, as in Eq. (49), remains to be answered. Such effects can be only understood, in general, from a detailed mathematical study of convergence of the method, which study is not pursued in the present paper and remains an open question. Assuming that the satisfaction of the LMB and AMB conditions in the manner of Eqs. (48 and 49) is "satisfactory", and further if the TBC condition is also met a priori, one can reduce Eq. (45) to a functional associated with the complementary energy principle,

$$\overset{*2}{\pi}_c (\dot{\underline{u}}, \dot{\underset{\sim}{s}}) = \int_{V_N} \left\{ - \dot{\underset{\sim}{S}}^* (\dot{\underset{\sim}{s}}) - 1/2\ \underset{\sim}{\tau}^N : [(\underline{\nabla}^N \dot{\underline{u}}) \cdot (\underline{\nabla}^N \dot{\underline{u}})^T] \right\} dV + \int_{S_{u_N}} \dot{\underline{t}} \cdot \dot{\bar{\underline{u}}}\ ds \tag{50}$$

In a finite element application, V_N can be subdivided into M subdomains, V_{mN} (m = 1,... M), each with a boundary ∂V_{mN}. In general it is seen that $\partial V_{mN} = S_{\sigma_{mN}} + S_{u_{mN}} + \rho_{mN}$ where $S_{\sigma_{mN}}$, $S_{u_{mN}}$ are, respectively, the portions of ∂V_{mN} where tractions and displacements are prescribed, and ρ_{mN} is that portion of ∂V_{mN} which is common to an adjacent element (interelement boundary). It is to be noted that in the finite element application of Eq. (50), the candidate stresses $\dot{\underset{\sim}{s}}$ should, not only, satisfy the LMB and AMB conditions, Eqs. (31) and (32), but also satisfy the interelement traction reciprocity condition†, $(\dot{\underline{t}})^+ + (\dot{\underline{t}})^- = 0$ (where $\dot{\underline{t}}$ is defined through Eq. (34)) at ρ_{mN} a priori. One can introduce this interelement condition directly as a condition of constraint into Eq. (50), in order to preserve a wide choice of candidate

† The superscripts (+) and (-) denote, respectively, the two sides of ρ_{mN} in the limit that ρ_{mN} is approached.

stress-rates $\dot{\underset{\sim}{s}}$, as:

$$\pi_{HS}^{*2}\,(\dot{\underline{u}},\dot{\underset{\sim}{s}},\dot{\tilde{\underline{u}}}_\rho) = \sum_m \int_{V_{mN}} \left\{ -\dot{S}^*(\dot{\underset{\sim}{s}}) - 1/2\ \underset{\sim}{\tau}^N : [(\underline{\nabla}^N\dot{\underline{u}})\cdot(\underline{\nabla}^N\dot{\underline{u}})^T]\right\} dV$$

$$+\sum_m \int_{S_{u_{mN}}} \dot{\underline{t}}\cdot\dot{\bar{\underline{u}}}\, ds + \sum_m \int_{\rho_{mN}} \dot{\underline{t}}\cdot\dot{\tilde{\underline{u}}}_\rho\, dS \qquad (51)$$

where $\dot{\tilde{\underline{u}}}_\rho$ are Lagrange-Multipliers to enforce the constraint of interelement traction reciprocity, and these can be seen to be the displacement rates at the interelement boundaries. The basic idea of choosing $\dot{\underset{\sim}{s}}$, as in Eqs. (47 and 48), to satisfy the LMB condition, and the modified complementary energy functional of Eq. (51), were used, in a somewhat less general fashion than is given here, by ATLURI (1973) in formulating a finite element method and applied to solve the problem of buckling of a shallow arch. The shortcomings of this approach were later discussed by ATLURI and MURAKAWA (1977). Moreover, in the modified complementary energy rate principle of Eq. (51), in addition to the fields $\dot{\underline{u}}$ and $\dot{\underset{\sim}{s}}$ within each element, the displacement-rate field at the interelement boundaries also enters as an independent variable. However in the finite element application of the Hellinger-Reissner type principle as in Eq. (45), only the two variables $\dot{\underset{\sim}{s}}$ (arbitrary, but differentiable and symmetric second order tensor) and $\dot{\underline{u}}$ (differentiable and interelement-compatible) need to be assumed within each element; and the interelement traction reciprocity condition then follows a posteriori from the variational principle. Considering this, and the fact that, in addition, one must study the convergence properties of the finite element scheme based on Eq. (51) to assess the effects of choosing the particular solution in a specific way as in Eqs. (48, 49), it appears most consistent and rational to directly apply the Hellinger-Reissner type principle as in Eq. (45). The finite element method thus generated based on assumed $\dot{\underset{\sim}{s}}$ as well as $\dot{\underline{u}}$, can be called a "mixed-method", which leads to simultaneous algebraic equations for finite element nodal displacements as well as nodal stresses (or alternatively, the nodal values of the stress functions $\underset{\sim}{A}$ as defined in Eq. (46)).

Finally, it is noted that the LMB condition may be satisfied more easily by choosing $\dot{\underset{\sim}{s}}$, unlike in Eqs. (46,47), such that,

$$\dot{\underset{\sim}{s}} = \text{curl curl } \underset{\sim}{A} - \underset{\sim}{\tau}^N \cdot (\underline{\nabla}^N \dot{\underline{u}}) + \dot{\underset{\sim}{s}}^P \tag{52}$$

where $\dot{\underset{\sim}{s}}^P$ is such that $\underline{\nabla}^N \cdot \dot{\underset{\sim}{s}}^P = -\rho^N \dot{\underline{B}}$. However the chosen $\dot{\underset{\sim}{s}}$ as in Eq. (52) then ceases to be symmetric, and thus the AMB condition must be introduced as a constraint condition, into the associated complementary energy functional of the type given in Eq. (50), through additional Lagrange Multipliers.

Thus it appears that a rate complementary energy principle in UL form based on $\dot{\underset{\sim}{s}}$ may not be consistent and practically useful in the analysis of finite strain plasticity problems.

3.1.2.2: In terms of $\dot{\underset{\sim}{t}}$, $\dot{\underset{\sim}{e}}$, and $\dot{\underline{u}}$

Analogous to the way discussed earlier, it can be shown that Eqs. (36, 37,39,41,35, and 19) follow as the Euler equations and natural boundary conditions corresponding to the stationarity of the functional,

$$\pi^{*2}_{HW}(\dot{\underline{u}},\dot{\underset{\sim}{e}},\dot{\underset{\sim}{t}}) = \int_{V_N} \left\{\dot{U}(\dot{\underset{\sim}{e}}) - \rho^N \dot{\underline{B}} \cdot \dot{\underline{u}} + \dot{\underset{\sim}{t}}^T : [(\underline{\nabla}^N \dot{\underline{u}})^T - \dot{\underset{\sim}{e}}]\right\} dV$$
$$- \int_{S_{\sigma_N}} \dot{\bar{\underline{t}}} \cdot \dot{\underline{u}} \, ds - \int_{S_{u_N}} \dot{\underline{t}} \cdot (\dot{\underline{u}} - \dot{\bar{\underline{u}}}) \, ds \tag{53}$$

where $\dot{U}$ is the rate potential for $\dot{\underset{\sim}{t}}$, as defined through Eqs. (20,21).

We now consider certain special cases of the above principle. If Eqs. (19,39, and 35) are met a priori, one can eliminate $\dot{\underset{\sim}{e}}$ and $\dot{\underset{\sim}{t}}$ as variables from Eq. (53) and derive a rate functional governing the rate potential energy principle, as:

$$\pi_p^{*2}(\dot{\underline{u}}) = \int_{V_N} \left\{\dot{U}(\dot{\underline{u}}) - \rho^N \dot{\underline{B}} \cdot \dot{\underline{u}}\right\} dV - \int_{S_{\sigma_N}} \dot{\underline{\bar{t}}} \cdot \dot{\underline{u}}\, ds \tag{54}$$

This rate variational principle was first stated by HILL (1959) and has been widely used in finite element applications to elastic-plastic problems [See for instance, NEEDLEMAN (1972), McMEEKING and RICE (1975), and NEMAT-NASSER and TAYA (1976)].

It is interesting to note that both the LMB and AMB conditions, Eqs. (36) and (37) respectively, as well as the TBC, Eq. (41), must follow from the principle, $\delta\pi_p^{*2}(\delta\dot{\underline{u}}) = 0$, with π_p^{*2} given as in Eq. (54). It is shown below that the AMB condition is inherently embedded in the special structure for $\dot{U}$. Thus, using the definition of $\dot{\underset{\sim}{t}}$ $(\equiv \partial\dot{U}/\partial\dot{\underset{\sim}{e}}^T)$ and Eq. (21), it is seen that,

$$\dot{\underset{\sim}{t}} = \frac{\partial \dot{U}}{\partial \dot{\underset{\sim}{e}}^T} \equiv \frac{\partial}{\partial \dot{\underset{\sim}{e}}^T} [\dot{W} + 1/2\, \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{e}}^T \cdot \dot{\underset{\sim}{e}})] \tag{55}$$

$$= \frac{\partial \dot{W}}{\partial \dot{\underset{\sim}{\epsilon}}} \frac{\partial \dot{\underset{\sim}{\epsilon}}}{\partial \dot{\underset{\sim}{e}}^T} + \underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{e}}^T \tag{56a}$$

$$= \dot{\underset{\sim}{s}} + \underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{e}}^T \tag{56b}$$

Wherein, the definition of $\dot{\underset{\sim}{s}}$ from Eq. (17) has been used. Substituting for $\dot{\underset{\sim}{t}}$ from Eq. (56b) into the AMB condition, Eq. (37), it is seen that the AMB condition is inherently met. This is due to the special structure for $\dot{U}$ as given through Eq. (21). Conversely, it is seen that if, instead of Eqs. (20, and 21), an arbitrary $\dot{U}$ is <u>postulated</u> as a function of $\dot{\underset{\sim}{e}}$, then the principle based on the functional in Eq. (54) ceases to be valid, since the AMB condition ceases either to be built into the structure of $\dot{U}$ or to follow unambiguously as an Euler equation from the vanishing of the first variation of the said functional. This fact appears to be never explicitly stated in the literature.

Now, by inverting the bi-linear relation in Eq. (19) one may achieve a contact transformation,

$$\dot{U} - \dot{\underset{\sim}{t}}^T : \dot{\underset{\sim}{e}} = -\dot{E}^*(\dot{\underset{\sim}{t}}) \tag{57}$$

such that,
$$\frac{\partial \dot{E}^*}{\partial \dot{\underset{\sim}{t}}} = \dot{\underset{\sim}{e}}^T \tag{58}$$

Using Eq. (57) to eliminate $\dot{\underset{\sim}{e}}$ from Eq. (53) one may formally obtain a functional:

$$\pi^{*2}_{HR}(\dot{\underset{\sim}{t}},\dot{\underline{u}}) = \int_{V_N} \left\{- \dot{E}^*(\dot{\underset{\sim}{t}}) - \rho_N \underline{B} \cdot \dot{\underline{u}} + \dot{\underset{\sim}{t}}^T : [(\underline{\nabla}^N \dot{\underline{u}})^T]\right\} dV$$

$$- \int_{S_{\sigma_N}} \dot{\underline{t}} \cdot \dot{\underline{u}} \, ds - \int_{S_{u_N}} \dot{\underline{t}} \cdot (\dot{\underline{u}} - \dot{\bar{\underline{u}}}) \, ds. \tag{59}$$

In fact the above functional, as the basis for a Hellinger-Reissner type variational principle was used by NEALE (1972). However, the validity of such a principle needs a closer examination. If $\delta\pi^{*2}_{HR} = o$, with π^{*2}_{HR} as given in Eq. (59), is a valid Hellinger-Reissner type rate principle, we note that the corresponding Euler equations and natural boundary conditions must be: (i) the LMB condition, Eq. (36); (ii) the AMB condition, Eq. (37); (iii) the compatibility condition, Eq. (39), (iv) TBC, Eq. (41); and (v) the DBC, Eq. (35).

It is seen upon examining Eqs. (17) to (25) of NEALE'S (1972) development, the AMB condition, Eq. (37) of the present paper, is in fact not an Euler equation of the principle $\delta I = 0$ [Eq. (22) of NEALE'S (1972) work, which is identical to $\delta\pi^{*2}_{HR} = 0$ of the present article]. Thus, if at all the AMB condition is satisfied, for the validity of the principle, this condition must be embedded in the special structure, if any, for the complementary energy density function, $\dot{E}^*(\dot{\underset{\sim}{t}})$ of Eq. (57) above. To examine this possibility, consider the form of $\dot{U}$ as given from Eqs. (20 and 16):

$$2\dot{U} = \dot{e}_{ij}L_{ijk\ell}\dot{e}_{k\ell} - 2\frac{\alpha}{g}(\lambda_{k\ell}\dot{e}_{k\ell})^2 - 2\tau^N_{ij}\dot{e}_{ik}\dot{e}_{kj} + \tau^N_{ij}\dot{u}_{k,i}\dot{u}_{k,j} \tag{60}$$

The stress rate, $\dot{\underset{\sim}{t}}$, as derived from the above, is

$$2\dot{t}_{ij} \equiv 2\frac{\partial \dot{U}}{\partial \dot{e}_{ji}} = (L_{ijk\ell} - 2\frac{\alpha}{g}\lambda_{ij}\lambda_{k\ell})\frac{(\dot{e}_{k\ell} + \dot{e}_{\ell k})}{2} - (\dot{e}_{im} + \dot{e}_{mi})\tau^N_{mj}$$

$$- \tau^N_{i\ell}(\dot{e}_{\ell j} - \dot{e}_{j\ell}) \tag{61}$$

where $L_{ijk\ell}$, α, g and $\lambda_{k\ell}$ are as defined before. The constitutive law, Eq. (61), is of bi-linear type for $\dot{t}_{ij}$ in terms of $\dot{e}_{k\ell}$. The inversion of this relation in <u>closed</u> <u>form</u>, to express $\dot{e}_{k\ell}$ in terms of $\dot{t}_{ij}$ appears to be impossible, in general.

Rearranging the right hand side of Eq. (61), it can be rewritten as:

$$\dot{t}_{ji} = {}^*L_{jikn}\,\dot{e}_{kn} \tag{62}$$

or matrix form $\{\dot{\underset{\sim}{t}}\}_{9x1} = [{}^*L]_{9x9}\,\{\dot{\underset{\sim}{e}}\}_{9x1}$ (62a)

Where ${}^*L_{ijk\ell}$ depends on the tensor of instantaneous elastic modulii $L_{ijk\ell}$ (which has the symmetry properties $L_{ijk\ell} = L_{jik\ell} = L_{ji\ell k} = L_{k\ell ij}$) and other quantities, α, g, λ_{ij} and τ^N_{ij}. However L^*_{jikn} has the only symmetry property,

$${}^*L_{jikn} = {}^*L_{knji} \tag{63}$$

Eventhough an analytical form for the inverse of the matrix $[{}^*L]$ of Eq. (62a) appears impossible to be obtained, one may numerically invert+ Eq. (62, 62a) to write,

+However, in the "first step" of the solution if the solid remains elastic, and the initial stresses $\underset{\sim}{\tau}^o$ are zero, in the initial configuration C_o, then ${}^*L_{ijk\ell} \equiv L_{ijk\ell}$ and hence the 9x9 matrix of Eq. (62a) cannot be inverted, because of the symmetry properties of $L_{ijk\ell}$ stated above.

$$\dot{e}_{ij} = {}^{*}L^{-1}_{ijk\ell}\ \dot{t}_{k\ell} \quad \text{or} \quad \{\dot{\underset{\sim}{e}}\} = [{}^{*}L]^{-1}\ \{\dot{\underset{\sim}{t}}\} \tag{64}$$

where, in general, ${}^{*}L^{-1}_{ijk\ell} = {}^{*}L^{-1}_{k\ell ij}$. Using Eq. (64) a contact transformation can in fact be made, to find $\dot{E}^{*}(\dot{\underset{\sim}{t}})$ such that

$$\partial\dot{E}^{*}/\partial\dot{t}_{ji} = \dot{e}_{ij} = {}^{*}L^{-1}_{ijk\ell}\ \dot{t}_{k\ell} \tag{65}$$

If the AMB condition, Eq. (37), is inherently embedded in the structure of $\dot{E}^{*}(\dot{\underset{\sim}{t}})$, then the condition,

$$\dot{e}_{ij}\ \tau^{N}_{jk} + \dot{t}_{ik} = \text{symmetric} \tag{66}$$

must be identically satisfied when $\dot{e}_{ij}$ is expressed in terms of $\dot{t}_{mn}$ through Eq. (65). Doing so we see that the AMB condition is expressed by the necessary condition on the structure of $\dot{E}^{*}(\dot{\underset{\sim}{t}})$ that

$$(\partial\dot{E}^{*}/\partial\dot{t}_{ji})\ \tau^{N}_{jk} + \dot{t}_{ik} \text{ must be symmetric}$$

$$\text{or} \quad \underbrace{{}^{*}L^{-1}_{ijmn}\ \dot{t}_{mn}\ \tau^{N}_{jk}}_{\mathrm{I}} + \underbrace{\dot{t}_{ik}}_{\mathrm{II}} \quad \text{must be symmetric} \tag{67}$$

It can be seen that neither of the two terms, I and II above, is by itself symmetric under $i \longleftrightarrow k$ interchange. The other possible ways in which I + II above can be symmetric under $i \longleftrightarrow k$ interchange are: (i) firstly, one term is a transpose of the other; however, it is easy to see that this is not the case; (b) secondly, the first term can be expressed as the sum of a symmetric tensor and the transpose of the second term. Since ${}^{*}L^{-1}_{ijk\ell}$ cannot be found analytically, and with the only knowledge that ${}^{*}L^{-1}_{ijk\ell} = {}^{*}L^{-1}_{k\ell ij}$, it appears impossible to verify this assertion.

Thus, since the AMB condition is neither clearly an Euler equation corresponding to the stationarity of π^{*2}_{HR} of Eq. (59), nor can be verified to be embedded in the structure of $\dot{E}^{*}(\dot{\underset{\sim}{t}})$, the Hellinger-Reissner type principle based on Eq. (59) appears to be of little practical value. For similar reasons, the complementary energy rate principle as stated by HILL (1959) (which can be derived formally from Eq. (59) by requiring $\dot{\underset{\sim}{t}}$ to satisfy the LMB condition, Eq. (36), and the TBC, Eq. (41), a priori) also appears to be of little practical value.

3.1.2.3: In terms of $\dot{\underset{\sim}{r}}(\dot{\underset{\sim}{t}},\dot{\underset{\sim}{\alpha}})$; $\dot{\underset{\sim}{\alpha}}(\equiv\dot{\underset{\sim}{\omega}})$; $\dot{\underset{\sim}{h}}$, $\dot{\underline{u}}$

To seek alternative ways to avoid the above discussed difficulties in formluating a consistent complementary energy rate principle and Hellinger-Reissner type rate principle, we transform the general variational principle associated with Eq. (53) into one involving $\dot{\underset{\sim}{r}}$, $\dot{\underset{\sim}{\alpha}}$, $\dot{\underset{\sim}{h}}$ and $\dot{\underline{u}}$ as variables.

First, by comparing Eqs. (20) and (28) we note that,

$$\dot{U} = \dot{Q} + 1/2\,\underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\varepsilon}}.\dot{\underset{\sim}{\varepsilon}}) - \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\varepsilon}}.\dot{\underset{\sim}{\varepsilon}}) + 1/2\,\underset{\sim}{\tau}^N : (\dot{\underset{\sim}{e}}^T.\dot{\underset{\sim}{e}})$$

$$= \dot{Q} + 1/2\,\underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\alpha}}^T.\dot{\underset{\sim}{\alpha}}) + \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\alpha}}^T.\dot{\underset{\sim}{\varepsilon}}) \tag{68}$$

Using (68) to express $\dot{U}$ in terms of $\dot{Q}$ (which is a function of $\dot{\underset{\sim}{h}} \equiv \dot{\underset{\sim}{\varepsilon}}$) and writing $\dot{\underset{\sim}{e}} = \dot{\underset{\sim}{h}} + \dot{\underset{\sim}{\alpha}}$, we rewrite Eq. (53) as:

$$\begin{aligned} \pi^{*2}_{HW}(\dot{\underline{u}};\dot{\underset{\sim}{h}};\dot{\underset{\sim}{\alpha}};\dot{\underset{\sim}{t}}) = \int_{V_N} \Big\{ & \dot{Q}(\dot{\underset{\sim}{h}}) + 1/2\,\underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\alpha}}^T.\dot{\underset{\sim}{\alpha}}) + \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\alpha}}^T.\dot{\underset{\sim}{h}}) \\ & - \rho^N \dot{\underline{B}}.\dot{\bar{\underline{u}}} + \dot{\underset{\sim}{t}}^T : [(\underline{\nabla}^N \dot{\underline{u}})^T - \dot{\underset{\sim}{h}} - \dot{\underset{\sim}{\alpha}}] \Big\} dV \\ & - \int_{S_{\sigma_N}} \dot{\bar{\underline{t}}}.\dot{\underline{u}}\, ds - \int_{S_{u_N}} \dot{\underline{t}}(\dot{\underline{u}} - \dot{\bar{\underline{u}}})\, ds. \end{aligned} \tag{69}$$

In the above $\dot{\underset{\sim}{\alpha}}$ is a <u>skew-symmetric</u> tensor. The condition of vanishing of the first variation of the above functional can now be written as:

$$\delta\pi^{*2}_{HW}(\delta\dot{\underline{u}};\ \delta\underset{\sim}{\dot{h}};\ \delta\underset{\sim}{\dot{\alpha}};\ \delta\underset{\sim}{\dot{t}}) = \int_{V_N} \Big\{\Big[\frac{\partial\dot{Q}}{\partial\underset{\sim}{\dot{h}}} - 1/2(\underset{\sim}{\dot{t}} + \underset{\sim}{\tau}^{N}\cdot\underset{\sim}{\dot{\alpha}} + \underset{\sim}{\dot{t}}^{T} + \underset{\sim}{\dot{\alpha}}^{T}\cdot\underset{\sim}{\tau}^{N})\Big]:\ \delta\underset{\sim}{\dot{h}}$$

$$+ [(\underline{\nabla}^{N}\dot{\underline{u}})^{T} - \underset{\sim}{\dot{\alpha}} - \underset{\sim}{\dot{h}}]:\delta\underset{\sim}{\dot{t}}^{T} - [\underset{\sim}{\tau}^{N}\cdot\underset{\sim}{\dot{\alpha}} + \underset{\sim}{\dot{h}}\cdot\underset{\sim}{\tau}^{N} + \underset{\sim}{\dot{t}}]:\ \delta\underset{\sim}{\dot{\alpha}}^{T} - [\underline{\nabla}^{N}\cdot\underset{\sim}{\dot{t}} + \rho^{N}\dot{\underline{B}}]\cdot\delta\dot{\underline{u}}\Big\}dV$$

$$- \int_{S_{\sigma_N}} (\bar{\dot{\underline{t}}} - \underline{n}^{*}\cdot\underset{\sim}{\dot{t}})\cdot\delta\dot{\underline{u}}\,ds - \int_{S_{u_N}} \delta\dot{\underline{t}}\cdot(\dot{\underline{u}} - \bar{\dot{\underline{u}}})\,ds = 0 \qquad (70)$$

Noting that $\underset{\sim}{\dot{\alpha}}$, and hence $\delta\underset{\sim}{\dot{\alpha}}$, are skew-symmetric tensors, it can be clearly seen that the Euler Equations and natural boundary conditions corresponding to Eq. (70) are: (i) the constitutive law, Eq. (27); (ii) the LMB condition, Eq. (36); (iii) the AMB condition, Eq. (38); (iv) the compatibility condition, Eq. (40); (v) the TBC, Eq. (41); and (vi) the DBC, Eq. (35).

One can now invert (even if numerically) the relation of Eq. (27) and achieve a contact transformation,

$$\dot{Q} - \underset{\sim}{\dot{r}} : \underset{\sim}{\dot{h}} = -\dot{R}^{*}(\underset{\sim}{\dot{r}}) \qquad (71a)$$

or

$$\dot{Q} - 1/2[\underset{\sim}{\dot{t}} + \underset{\sim}{\dot{t}}^{T} + \underset{\sim}{\tau}^{N}\cdot\underset{\sim}{\dot{\alpha}} + \underset{\sim}{\dot{\alpha}}^{T}\cdot\underset{\sim}{\tau}^{N}]:\ \underset{\sim}{\dot{h}} = -\dot{R}^{*}(\underset{\sim}{\dot{r}}) \qquad (71b)$$

such that

$$\partial\dot{R}^{*}/\partial\underset{\sim}{\dot{r}} = \underset{\sim}{\dot{h}} \qquad (72)$$

Substituting Eq. (71) in Eq. (69) we can derive a functional, involving only $\dot{\underline{u}}$, and $\underset{\sim}{\dot{r}}$ (and hence $\underset{\sim}{\dot{t}}$ and $\underset{\sim}{\dot{\alpha}}$) as variables, corresponding to a Hellinger-Reissner type principle which has as its Euler Equations and natural b.c, Eqs. (36, 38, 40, 41, and 35). If, in addition, one assumes that the LMB condition and TBC for $\underset{\sim}{\dot{t}}$, Eqs. (36, and 41) are satisfied a priori, one can eliminate $\underset{\sim}{\dot{h}}$ and $\dot{\underline{u}}$ as variables from the functional in Eq. (69) and thus obtain a complementary energy functional:

$$\pi_C^{*2}(\dot{\underset{\sim}{\alpha}},\dot{\underset{\sim}{t}}) = \int_{V_N} \left\{- \dot{R}^*(\dot{\underset{\sim}{r}}) + 1/2\, \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\alpha}}^T \cdot \dot{\underset{\sim}{\alpha}}) - \dot{\underset{\sim}{t}}^T : \dot{\underset{\sim}{\alpha}}\right\} dV + \int_{S_{u_N}} (\underline{n}^* \cdot \dot{\underset{\sim}{t}}) \cdot \dot{\bar{\underline{u}}}\, ds \tag{73}$$

In the above, the definition of $\dot{\underset{\sim}{r}}$ [$\equiv 1/2\, (\dot{\underset{\sim}{t}} + \dot{\underset{\sim}{t}}^T + \underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{\alpha}} + \dot{\underset{\sim}{\alpha}}^T \cdot \underset{\sim}{\tau}^N)$] is implied; and the spin-rate field $\dot{\underset{\sim}{\alpha}}$ is required to be skew-symmetric. The variational equation, $\delta\pi_C^{*2} = 0$, for constrained $\delta\dot{\underset{\sim}{t}}$ (which obey the constraint $\underline{\nabla}^N \cdot \delta\dot{\underset{\sim}{t}} = 0$ in V_N and $\underline{n}^* \cdot \delta\dot{\underset{\sim}{t}} = 0$ at S_{σ_N}) and for constrained $\delta\dot{\underset{\sim}{\alpha}}$ (which is required to be skew symmetric) is seen to lead to

$$\delta\pi_C^{*2}(\delta\dot{\underset{\sim}{t}},\delta\dot{\underset{\sim}{\alpha}}) = 0 = \int \left\{\left[(\underline{\nabla}^N \dot{\underline{u}})^T - \dot{\underset{\sim}{\alpha}} - \frac{\partial \dot{R}^*}{\partial \dot{\underset{\sim}{r}}} \right] : \delta\dot{\underset{\sim}{t}}^T - [\dot{\underset{\sim}{t}} + \dot{\underset{\sim}{h}} \cdot \underset{\sim}{\tau}^N + \underset{\sim}{\tau}^N \cdot \dot{\underset{\sim}{\alpha}}] :$$

$$\delta\dot{\alpha}^T\Big\} dV + \int_{S_{u_N}} (\underline{n}^* \cdot \delta\dot{\underset{\sim}{t}})\, (\dot{\bar{\underline{u}}} - \dot{\underline{u}})\, dS. \tag{74}$$

Noting that by definition, $\partial\dot{R}^*/\partial\dot{\underset{\sim}{r}} \equiv \dot{\underset{\sim}{h}}$, it is clearly seen that Eq. (74) leads, as its Euler equations and natural b.c, (i) the compatibility condition, Eq. (40); (ii) the AMB condition, Eq. (38); and (iii) the DBC, Eq. (35).

Thus, Eq. (73) forms the basis of the most consistent and practically useful rate complementary energy theorem for the UL rate formulation of finite strain plasticity analysis methods because: (a) the admissible $\dot{\underset{\sim}{t}}$ is required to satisfy, a priori, only the uncoupled, linear LMB equation, Eq. (36), and TBC, Eq. (41), which can be met easily in applications, by setting $\dot{\underset{\sim}{t}} = \underline{\nabla}^N \times \underset{\sim}{\Psi} + \dot{\underset{\sim}{t}}^P$ where $\underset{\sim}{\Psi}$ are first order stress functions (once-differentiable) and $\dot{\underset{\sim}{t}}^P$ is any particular solution such that $\underline{\nabla}^N \cdot \dot{\underset{\sim}{t}}^P = -\rho^N \dot{\underline{B}}$; (b) the AMB conditions, the compatibility condition, and the DBC follow unambiguously as Euler equations.

In a finite element application of the complementary rate principle as stated through Eq. (73), the assumed stress-rate field $\dot{\underset{\sim}{t}}$ must not only satisfy the LMB condition (Eq. 36) within each element, but must also satisfy the traction reciprocity relation at the interelement boundary viz., $(\underline{n}^* . \dot{\underset{\sim}{t}})^+ + (\underline{n}^* . \dot{\underset{\sim}{t}})^-$ at ρ_{mN} [where + and -, respectively, indicate the two sides of ρ_{mN} in the limit that ρ_{mN} is approached]. This may, in general, pose a severe restriction on the choice of $\dot{\underset{\sim}{t}}$ within each element, especially when the element is of a arbitrary curved geometry. In such a case, it may be preferrable to include this interelement traction reciprocity condition as a constraint condition directly into the functional of Eq. (73). The Lagrange multipliers introduced to this end can be seen to be the interelement boundary displacements. The thus modified complementary energy rate principle for an assembly of a finite number of elements can be stated as the stationarity condition of the functional:

$$\pi_{MC}^{*2}(\dot{\underset{\sim}{\alpha}}, \dot{\underset{\sim}{t}}, \dot{\tilde{\underline{u}}}_\rho) = \sum_m \left\{ \int_{V_{mN}} \left[- \dot{R}^*(\dot{\underset{\sim}{r}}) + 1/2\, \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\alpha}}^T . \dot{\underset{\sim}{\alpha}}) - \dot{\underset{\sim}{t}}^T : \dot{\underset{\sim}{\alpha}} \right] dV \right.$$

$$\left. + \int_{S_{u_{mN}}} (\underline{n}^* . \dot{\underset{\sim}{t}}) . \dot{\bar{\underline{u}}}\, ds + \int_{\rho_{mN}} (\underline{n}^* . \dot{\underset{\sim}{t}}) . \dot{\tilde{\underline{u}}}_\rho\, ds \right\} \qquad (75)$$

In the above functional, $\dot{\underset{\sim}{\alpha}}$ and $\dot{\underset{\sim}{t}}$ are chosen independently within each element in terms of undetermined parameters, whereas $\dot{\tilde{\underline{u}}}_\rho$ are chosen in terms of displacements at nodes of a finite element and hence $\dot{\tilde{\underline{u}}}_\rho$ are common to elements sharing a common boundary. Thus the undetermined parameters in the field functions for $\dot{\underset{\sim}{\alpha}}$ and $\dot{\underset{\sim}{t}}$ can be eliminated at the element level and expressed in terms of the generalized nodal displacement coordinates. The finite element method based on Eq. (75) thus, in the end, results in a standard stiffness matrix procedure [See MURAKAWA and ATLURI (1978a,b) for instance, for details of finite element application of the complementary energy rate principles in finite elasticity]. Alternatively, the interelement traction reciprocity can be satisfied a priori by an appropriate choice of the first-order stress functions $\underset{\sim}{\Psi}$ from which the equilibrated $\dot{\underset{\sim}{t}}$ are derived.

The finite element method then, in general, will lead to a "flexibility matrix" type approach. Such "stiffness" and "flexibility" type finite element methods based on Eqs. (75) and (73), respectively, for analyzing certain metal-forming problems are the subjects of the author's work in progress and will be subjects of a forthcoming paper.

3.1.3: Near Incompressibility in the Fully Plastic Range:

As discussed in the introduction, an important aspect of numerical schemes for finite strain elastic-plastic analysis is the problem of accurate treatment of nearly incompressible deformation rates at such magnitudes of strain. In a finite element application of the potential energy rate formulation of the type given by Eqs. (43 or 54), if the assumed deformation rates do not a priori obey the incompressibility constraint, it may be necessary to retain this constraint as an a posteriori constraint through a Lagrange multiplier, the hydrostatic pressure. To this end, consider the rate potential $\dot{W}(\dot{\underset{\sim}{\epsilon}})$, Eq. (18), for a classical Prandtl-Reuss type rate constitutive law:

$$\dot{W}(\dot{\underset{\sim}{\epsilon}}) = 1/2\, \dot{\underset{\sim}{\sigma}}^{*} : \dot{\underset{\sim}{\epsilon}} - \underset{\sim}{\tau}^{N} : (\dot{\underset{\sim}{\epsilon}} \cdot \dot{\underset{\sim}{\epsilon}}) \tag{76}$$

The corotational rate of Kirchhoff stress, $\dot{\underset{\sim}{\sigma}}^{*}$, for a classical Prandtl-Reuss type approximation can be written in terms of $\dot{\underset{\sim}{\epsilon}}$, as suggested by McMEEKING and RICE (1975), as

$$\dot{\sigma}^{*}_{ij} = 2\mu\left[\delta_{ik}\delta_{j\ell} - \frac{9\alpha\mu}{(2h + 6\mu)} \frac{\tau'^{N}_{ij}\tau'^{N}_{k\ell}}{(\bar{\tau}^{N})^{2}}\right]\dot{\epsilon}_{k\ell} + \lambda\dot{\epsilon}_{kk}\delta_{ij} \tag{77}$$

where, $\alpha = 1$ if at yield and $\dot{\tau}'_{ij}\dot{\epsilon}_{ij} > 0$ and $\alpha = 0$ otherwise; τ'^{N}_{ij} is the deviatoric Cauchy stress in C_N; $(\bar{\tau}^{N})^{2} = (3/2)\tau'_{ij}\tau'_{ij}$; h is the slope of the uniaxial stress/plastic strain curve; and λ and μ, are Lame's constants. We rewrite Eq. (77) as:

$$\dot{\sigma}^*_{ij} = 2\mu\, E_{ijk\ell}\dot{\epsilon}_{k\ell} + \lambda\dot{\epsilon}_{kk}\delta_{ij} \tag{78}$$

where the definition of $E_{ijk\ell}$ is apparent from comparing Eqs. (77) and (78). We can write $\dot{\underset{\sim}{\sigma}}^*$ in terms of its deviatoric and hydrostatic parts, as : $\dot{\sigma}^*_{ij} = \dot{\sigma}^{*\prime}_{ij} + (1/3)\,\dot{\sigma}^*_{kk}\delta_{ij}$; and likewise, express $\dot{\epsilon}_{ij} = \dot{\epsilon}'_{ij} + (1/3)\dot{\epsilon}_{kk}\delta_{ij}$. From Eq. (78) one can write;

$$\dot{\sigma}^{*\prime}_{ij} = 2\mu\, E_{ijk\ell}\dot{\epsilon}'_{k\ell} \quad ; \quad \dot{\sigma}^*_{kk} = (3\lambda + 2\mu)\dot{\epsilon}_{kk} \tag{79a,b}$$

Using Eqs. (77-79a,b) in Eq. (76) we can write,

$$\dot{W} = \mu E_{ijk\ell}\dot{\epsilon}_{k\ell}\dot{\epsilon}_{ij} + \frac{\lambda}{2}(\dot{\epsilon}_{kk})^2 - \underset{\sim}{\tau}^N : (\dot{\underset{\sim}{\epsilon}} \cdot \dot{\underset{\sim}{\epsilon}}) \tag{80}$$

or, equivalently,

$$\dot{W} = \mu E_{ijk\ell}\dot{\epsilon}'_{k\ell}\dot{\epsilon}'_{ij} + \left(\frac{3\lambda + 2\mu}{6}\right)(\dot{\epsilon}_{kk})^2 - \tau^N : (\dot{\underset{\sim}{\epsilon}} \cdot \dot{\underset{\sim}{\epsilon}}) \tag{81}$$

To obtain numerically accurate solutions in situations of fully developed plasticity it may be advantageous to retain the hydrostatic pressure as an independent variable, and thus derive a mixed variational principle which represents a modification to the potential energy rate principle given through Eq. (43), with $\dot{W}$ expressed as in Eq. (80). To this end, we introduce both $\dot{\epsilon}_{ij}$ and $\dot{\epsilon}_{kk}$ as additional independent variables into the functional of Eq. (43), through the introduction of Lagrange multipliers β_{ij} and α, respectively. From this general variational principle (with u, $\dot{\epsilon}_{ij}$, $\dot{\epsilon}_{kk}$, $\dot{\beta}_{ij}$, and α, all as variables) we demand all the necessary field equations in the case of near incompressibility. When the Lagrange multipliers are identified with the relevant stress rates, this general rate variational principle can be written as:

$$\pi_{MW}^{2*}(\underline{\dot{u}};\ \dot{\epsilon}_{ij};\ \dot{\epsilon}_{kk};\ \dot{\sigma}_{ij}^{*};\ \dot{\sigma}_{kk}^{*})$$

$$= \int_{V_N} \{\mu\ E_{ijk\ell}\dot{\epsilon}_{k\ell}\dot{\epsilon}_{ij} + \frac{\lambda}{2}(\dot{\epsilon}_{kk})^2 + [\dot{u}_{k,j}\delta_{kj} - \dot{\epsilon}_{kk}]\ \frac{\lambda\dot{\sigma}_{kk}^{*}}{(3\lambda + 2\mu)})$$

$$+ [\dot{u}_{(i,j)} - \dot{\epsilon}_{ij}][\dot{\sigma}_{ij}^{*} - \frac{\lambda\dot{\sigma}_{kk}^{*}\delta_{ij}}{3\lambda + 2\mu}] - \tau_{ij}^{N}\dot{u}_{(k,i)}\dot{u}_{(k,j)}$$

$$+ 1/2\ \tau_{ij}^{N}\dot{u}_{k,i}\dot{u}_{k,j} - \rho^{N}\dot{B}_{i}\dot{u}_{i}\Big\}\ dV + \text{Boundary terms} \qquad (82)$$

The above general principle can be seen to be valid in both the cases of near and precise incompressibility. In the above, the notation, $\dot{u}_{(k,i)} = 1/2(\dot{u}_{k,i} + \dot{u}_{i,k})$, and $\dot{u}_{i,k} = \partial\dot{u}_i/\partial y_k^N$ has been used. If from Eq. (82) one eliminates (i) $\dot{\epsilon}_{ij}$ by defining a priori $\dot{\epsilon}_{ij} = \dot{u}_{(i,j)}$, and (ii) eliminate $\dot{\epsilon}_{kk}$ through a contact transformation:

$$\frac{\lambda}{2}\ (\dot{\epsilon}_{kk})^2 - \dot{\epsilon}_{kk}\ \frac{\lambda\dot{\sigma}_{kk}^{*}}{(3\lambda + 2\mu)} = - \frac{\lambda}{2}\ \frac{(\dot{\sigma}_{kk}^{*})^2}{(3\lambda + 2\mu)^2} \qquad (83)$$

one obtains a mixed variational principle, involving $\dot{u}_i$ and $\dot{\sigma}_{kk}^{*}$ as variables, governed by the functional,

$$\pi_{mp1}^{*2}\ (\dot{u}_i;\ \dot{\sigma}_{kk}^{*}) = \int_{V_N} \{\mu E_{ijk\ell}\dot{u}_{(k,\ell)}\dot{u}_{(i,j)} - \frac{\lambda}{2}\ \frac{(\dot{\sigma}_{kk}^{*})^2}{(3\lambda + 2\mu)^2}$$

$$+ \frac{\lambda}{(3\lambda + 2\mu)}\ \dot{\sigma}_{kk}^{*}\dot{u}_{\ell,j}\delta_{\ell j} - \tau_{ij}^{N}\dot{u}_{(k,i)}\dot{u}_{(k,j)}$$

$$+ \frac{1}{2}\ \tau_{ij}^{N}\dot{u}_{k,i}\dot{u}_{k,j} - \rho^{N}\dot{B}_{i}\dot{u}_{i}\}\ dV - \int_{S_T} \dot{\bar{t}}_{i}\dot{u}_{i}ds \qquad (84)$$

which remains valid for nearly or even precisely incompressible behavior at large plastic strains for all assumed displacement rates $\dot{u}_i$ that do not obey the constraint of incompressibility a priori. Eq. (84) and the associated variational principle are analogous to the ones in the case of linear isotropic elasticity given directly by HERRMANN (1965); however, the way in which HERRMANN arrived at his principle for the linear elastic infinitesimal deformation case is not evident from reading his work.

Likewise, using the definition of $\dot{W}$ as in Eq. (81) in Eq. (43), and introducing $\dot{\epsilon}'_{k\ell}$ and $\dot{\epsilon}_{kk}$ as additional independent variables into the functional in Eq. (43) through appropriate Lagrange Multipliers, one can derive another alternate general variational principle, which remains valid in the limit of incompressibility, with the associated functional:

$$\pi^{*2}_{HW}(\dot{u}_i; \dot{\epsilon}'_{ij}; \dot{\epsilon}_{kk}; \dot{\sigma}^{*\prime}_{ij}; \dot{\sigma}^{*}_{kk})$$

$$= \int_{V_N} \Big\{\mu E_{ijk\ell}\dot{\epsilon}'_{ij}\dot{\epsilon}'_{k\ell} + \frac{3\lambda + 2\mu}{6}\dot{\epsilon}^2_{kk} + [\dot{u}_{k,k} - \dot{\epsilon}_{kk}]\frac{\dot{\sigma}^*_{kk}}{3}$$

$$+ [\dot{u}'_{(i,j)} - \dot{\epsilon}'_{ij}]\,\dot{\sigma}^{*\prime}_{ij} - \tau^N_{ij}\dot{u}_{(k,i)}\dot{u}_{(k,j)} + \frac{1}{2}\tau^N_{ij}\dot{u}_{k,i}\dot{u}_{k,j}$$

$$- \rho_N\dot{B}_i\dot{u}_i\Big\}\, dV + \text{boundary terms} \tag{85}$$

where $\dot{u}'_{(i,j)} = \dot{u}_{(i,j)} - \dot{u}_{k,k}\delta_{ij}/3$ and $\dot{u}_{(i,j)}$ is defined earlier. If from Eq. (85) one eliminates $\dot{\epsilon}'_{ij}$ as a variable through a priori satisfying the condition $\dot{\epsilon}'_{ij} = \dot{u}'_{(i,j)}$; and $\dot{\epsilon}_{kk}$ is eliminated through the contact transformation given below,

$$\left(\frac{3\lambda + 2\mu}{6}\right)\dot{\epsilon}^2_{kk} - \frac{\dot{\epsilon}_{kk}\dot{\sigma}^*_{kk}}{3} = - \frac{\dot{\sigma}^2_{kk}}{6(3\lambda + 2\mu)} \tag{86}$$

one obtains an alternate mixed variational principle, also involving $\dot{u}_i$ and $\dot{\sigma}^*_{kk}$ as variables, governed by the functional:

$$\pi^{*2}_{mp2}(\dot{u}_i; \dot{\sigma}^*_{kk}) = \int_{V_N}\Big\{\mu E_{ijk\ell}\dot{u}'_{(i,j)}\dot{u}'_{(k,\ell)} + \frac{\dot{\sigma}^*_{kk}}{3}\dot{u}_{k,k}$$

$$\frac{-(\dot{\sigma}^*_{kk})^2}{6(3\lambda + 2\mu)} - \tau^N_{ij}\dot{u}_{(k,i)}\dot{u}_{(k,j)} + \frac{1}{2}\tau^N_{ij}\dot{u}_{k,i}\dot{u}_{k,j} - \rho_N\dot{B}_i\dot{u}_i\Big\}dV$$

$$- \int_{s_T}\dot{\bar{t}}_i\dot{u}_i\, ds \tag{87}$$

Equation (87) and the associated principle are analogous to those derived by KEY (1969) for linear elastic infinitesmal deformation problems; except that, KEY (1969), derives such a principle through FRAEIJS DE VEUBEKE'S (1951) interpretation of what is generally known as the Hellinger-Reissner theorem in linear elasticity.

NAGTEGAAL, PARKS, and RICE (1974, Appendix II therein), to improve the accuracy of UL rate finite element formulations for problems of large plastic flow, suggest a mixed formulation based on the functional:

$$\pi^{*2}_{mp3}(\dot{u}_i;\dot{\epsilon}_{kk}) = \int_{V_N} \left\{\frac{1}{2}\,\dot{\sigma}^{*\prime}_{ij}\dot{\epsilon}'_{ij} + \frac{(3\lambda + 2\mu)}{3}\left(\dot{\sigma}^{*}_{kk}\dot{\epsilon}_{kk} - \frac{1}{2}\dot{\epsilon}^{2}_{kk}\right)\right.$$

$$\left. - \tau^{N}_{ij}\dot{u}_{(k,i)}\dot{u}_{(k,j)} + \frac{1}{2}\tau^{N}_{ij}\dot{u}_{k,i}\dot{u}_{k,j} - \rho_N\dot{B}_i\dot{u}_i\right\}dV - \int_{s_T}\dot{\bar{t}}_i\dot{u}_i ds \qquad (88)$$

where $\dot{\epsilon}'_{ij} = \dot{u}'_{(i,j)}$ and $\dot{\sigma}^{*\prime}_{ij}$ is related to $\dot{\epsilon}'_{ij}$ through an equation of the type of Eq. (79a). It is worth noting that the above formulation, Eq. (88), is analogous to the present formulation given in Eq. (87), except for the fact that, whereas $\dot{\sigma}^{*}_{kk}$ appears as a variable in Eq. (87), $\dot{\epsilon}_{kk}$ appears in Eq. (88). It is interesting to observe that the procedure based on Eq. (88) ceases to be valid in the limit of precise incompressibility. Moreover, in the discrete (finite-element) version of the functional corresponding to Eq. (88) (when appropriate discrete approximations for $\dot{u}_i$ and $\dot{\epsilon}_{kk}$ are introduced), NAGTEGAAL, et al., (1974) proceed to eliminate $\dot{\epsilon}_{kk}$ as a variable at the element level and introduce a modified definition for the strain energy density functional, $\dot{W}$. The rigorous theoretical validity of the modified discrete functional, as a variational basis for obtaining discretized equilibrium equations, appears somewhat questionable.

We note that the above discussed difficulties with the incompressibility constraint are somewhat easier to handle in the case of assumed stress finite element methods based on a complementary rate principle of type given in Eqs. (73 and 74). [For a treatment of incompressibility, using assumed stress finite element methods, see for instance, the works of TONG (1969), and PIAN and LEE (1976) in linear elastic infenitesual deformation cases and that of MURAKAWA and ATLURI (1978b) in finite elasticity problems].

3.2: Rate Variational Principles in Total Lagrangean Formulation:

In the numerical solution of certain problems such as, for instance, plates and shells, it may be preferrable to use rate formulations wherein all the variables in each subsequent increment are referred to a fixed Lagrangean or Total Lagrangean (TL) frame. Thus in the TL formulation, the initial configuration C_o, with coordinates x_i, is used to refer all the state variables in each of the subsequent configurations. Let $\underset{\sim}{s}'$ and $\underset{\sim}{t}'$ be the rates of 2nd and 1st Piola-Kirchhoff stresses, in going from C_N to C_{N+1}, which stress rates are referred to and measured per unit area in the initial configuration C_o. Let $\underline{\nabla}^o$ be the gradient operator in the coordinates in C_o, and set $\dot{\underline{u}}$ be the rate of displacement from the current state. Then the Total Lagrangean strain-rate, $\underset{\sim}{E}'$, is given by,

$$\underset{\sim}{E}' = \frac{1}{2}\left[\underline{\nabla}^o\dot{\underline{u}} + (\underline{\nabla}^o\dot{\underline{u}})^T + (\underline{\nabla}^o\dot{\underline{u}}) \cdot (\underline{\nabla}^o\underline{u}^N)^T + (\underline{\nabla}^o\underline{u}^N) \cdot (\underline{\nabla}^o\dot{\underline{u}})^T\right] \tag{89}$$

where $\underline{u}^N$ is the displacement at C_N as measured from C_o. It is seen that the TL and UL strain-rates are related by:

$$\underset{\sim}{E}' = (\underset{\sim}{F}^N)^T \cdot \dot{\underset{\sim}{\varepsilon}} \cdot \underset{\sim}{F}^N. \tag{90}$$

where $\underset{\sim}{F}^N = (\underset{\sim}{I} + \underline{\nabla}^o\underline{u}^N)^T$ and $\dot{\underset{\sim}{\varepsilon}}$ is defined in Eq. (33). Likewise, if $\underset{\sim}{e}'$ $[\equiv(\underline{\nabla}^o\dot{\underline{u}})^T]$ is the TL rate of displacement gradient, it is related to the UL rate $\dot{\underset{\sim}{e}}$ by

$$\underset{\sim}{e}' = \dot{\underset{\sim}{e}} \cdot \underset{\sim}{F}^N \tag{91}$$

Also, from Eqs. (3 and 2), respectively, the relations between the TL rates $\underset{\sim}{s}'$ and $\underset{\sim}{t}'$ and UL rates $\dot{\underset{\sim}{s}}$ and $\dot{\underset{\sim}{t}}$, respectively, can be derived as:

$$\underset{\sim}{s}' = J^N (\underset{\sim}{F}^N)^{-1} \dot{\underset{\sim}{s}} (\underset{\sim}{F}^N)^{-T} \tag{92}$$

and
$$\underset{\sim}{t}' = J^N (\underset{\sim}{F}^N)^{-1} \dot{\underset{\sim}{t}} = \underset{\sim}{s}' \cdot \underset{\sim}{F}^{NT} + \underset{\sim}{s}^N \cdot \underset{\sim}{e}'^T \tag{93a,b}$$

where J^N is the value of the determinant of the matrix $[y^N_{i,j}]$. Finally, the TL rate of Jaumann stress, $\underset{\sim}{r}'$, is seen, from Eqs.(4) and (5), to be related to $\underset{\sim}{t}'$ and $\underset{\sim}{s}'$ by:

$$\underset{\sim}{r}' = \frac{1}{2} [\underset{\sim}{t}^N \cdot \underset{\sim}{\alpha}' + \underset{\sim}{\alpha}'^T \cdot \underset{\sim}{t}^{NT} + \underset{\sim}{t}' \cdot \underset{\sim}{\alpha}^N + \underset{\sim}{\alpha}^{NT} \cdot \underset{\sim}{t}'^T] \tag{94}$$

$$= \frac{1}{2} [\underset{\sim}{s}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) + (\underset{\sim}{I} + \underset{\sim}{h}^N) \cdot \underset{\sim}{s}' + \underset{\sim}{s}^N \cdot \underset{\sim}{h}' + \underset{\sim}{h}' \cdot \underset{\sim}{s}^N] \tag{95}$$

where $\underset{\sim}{t}^N$ and $\underset{\sim}{s}^N$ are, respectively, the first and second Piola-Kirchhoff stress tensors in C_N as referred to and measured per unit area in C_o; $\underset{\sim}{h}^N$ is the engineering strain tensor in C_N referred to C_o; and $\underset{\sim}{\alpha}^N$ and $\underset{\sim}{\alpha}'$ are rotation tensors, such that $(\underset{\sim}{\alpha}^N + \underset{\sim}{\alpha}')$ is an orthogonal tensor, and these are found from the application of the polar-decomposition theorem as:

$$(\underline{\nabla}^o \underline{y}^N)^T = \underset{\sim}{\alpha}^N \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) \text{ and } (\underline{\nabla}^o \dot{\underline{u}})^T = \underset{\sim}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) + \underset{\sim}{\alpha}^N \cdot \underset{\sim}{h}' \tag{96}$$

Now we consider the question of the forms of rate potentials for $\underset{\sim}{s}'$, $\underset{\sim}{t}'$, and $\underset{\sim}{r}'$. First, we note that if a rate potential for $\dot{\underset{\sim}{s}}$ of the form of Eq. (17) exists, then, inview of Eqs. (90) and (92), a rate potential say W', can also be shown to exist for $\underset{\sim}{s}'$. Specifically, let the potential $\dot{W}$ for $\dot{\underset{\sim}{s}}$ Eq. (17) be written as:

$$\dot{W} = \frac{1}{2} M_{ijk\ell} \dot{\epsilon}_{ij} \dot{\epsilon}_{k\ell}; \ \partial \dot{W}/\partial \dot{\underset{\sim}{\epsilon}} = \dot{\underset{\sim}{s}} \tag{97a,b}$$

Where, the tensor $M_{ijk\ell}$ can be expressed interms of $L_{ijk\ell}$ and the relevant plasticity parameters through Eqs. (18 and 16).

Then, in view of Eqs. (90) and (92), the rate potential W' for $\underset{\sim}{s}'$ can be written as:

$$W' = \frac{1}{2} M'_{ijk\ell} \, E'_{ij} \, E'_{k\ell}; \qquad \partial W'/\partial \underset{\sim}{E}' = \underset{\sim}{s}' \tag{98a,b}$$

$$\text{where} \quad M'_{ijk\ell} = J^N M_{rstp} \frac{\partial x_i}{\partial y^N_r} \frac{\partial x_j}{\partial y^N_s} \frac{\partial x_k}{\partial y^N_t} \frac{\partial x_\ell}{\partial y^N_p} \tag{99}$$

Likewise, inview of relation (93b) it can be seen that a rate potential U' exists for $\underset{\sim}{t}'$, for the case of an classical elasto-plastic material, where,

$$U' = W' + \frac{1}{2} \underset{\sim}{s}^N : (\underset{\sim}{e}'^T \cdot \underset{\sim}{e}'); \; \partial U'/\partial \underset{\sim}{e}'^T = \underset{\sim}{t}' \tag{100a,b}$$

In writing the right hand side of Eq. (100a), the relation $\underset{\sim}{E}' = 1/2$ $(\underset{\sim}{e}'^T \cdot \underset{\sim}{F}^N + \underset{\sim}{F}^{NT} \cdot \underset{\sim}{e}')$ may be used. Likewise, inview of Eq. (95), it can also be seen that a rate potential Q' for the TL rate of Jaumann stress, $\underset{\sim}{r}'$, also exists, where,

$$Q' = W' + \frac{1}{2} \underset{\sim}{s}^N : (\underset{\sim}{h}' \cdot \underset{\sim}{h}'); \; \partial Q'/\partial \underset{\sim}{h}' = \underset{\sim}{r}' \tag{101a,b}$$

Again, in writing the right hand side of Eq. (101a), the relation, $\underset{\sim}{E}' =$ $1/2 \, [\underset{\sim}{h}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) + (\underset{\sim}{I} + \underset{\sim}{h}^N) \cdot \underset{\sim}{h}']$ may be used.

Now we consider the rate form of the field equations and boundary conditions. Considering the rates of Eqs. (6-12c) these can be written as:

<u>In terms of $\underset{\sim}{s}'$; $\underset{\sim}{E}'$; and $\underline{u}'$:</u>

$$\text{(LMB)}\rightarrow \qquad \underline{\nabla}^o \cdot \left\{ \underset{\sim}{s}^N \cdot \underset{\sim}{e}'^T + \underset{\sim}{s}' \cdot (\underset{\sim}{F}^N)^T \right\} + \rho^o \underline{B}' = 0 \tag{102}$$

$$\text{(AMB)}\rightarrow \qquad \underset{\sim}{s}' = \underset{\sim}{s}'^T \tag{103}$$

(Compatibility) $\underset{\sim}{E}' = \frac{1}{2}[\underset{\sim}{e}' + \underset{\sim}{e}'^T + \underset{\sim}{e}'^T \cdot \underset{\sim}{e}^N + \underset{\sim}{e}^{NT} \cdot \underset{\sim}{e}']$ (104)

where $\underset{\sim}{e}' = (\underline{\nabla}^o \underline{u})^T$

(TBC)→ $\underline{n} \cdot \{\underset{\sim}{s}^N \cdot \underset{\sim}{e}'^T + \underset{\sim}{s}' \cdot \underset{\sim}{F}^{NT}\} \equiv \underline{t}' = \underline{\bar{t}}'$ on S_{σ_o} (105)

Where $\underline{\bar{t}}'$ are prescribed tractions per unit area of the surface segment S_{σ_o} of the boundary of the solid in C_o, and $\underline{n}$ is a unit outward normal to S_{σ_o}.

(DBC)→ $\underline{\dot{u}} = \underline{\dot{\bar{u}}}$ at S_{u_o} (106)

In terms of $\underset{\sim}{t}'$; $\underset{\sim}{e}'$; and $\underline{\dot{u}}$ (or $\underset{\sim}{r}'$, $\underset{\sim}{\alpha}'$, $\underset{\sim}{h}'$, and $\underline{\dot{u}}$):

(LMB)→ $\underline{\nabla}^o \cdot \underset{\sim}{t}' + \rho^o \underline{B}' = 0$ (107)

(AMB)→ $\underset{\sim}{F}^N \cdot \underset{\sim}{t}' + \underset{\sim}{e}' \cdot \underset{\sim}{t}^N = \underset{\sim}{t}^{NT} \cdot \underset{\sim}{e}'^T + \underset{\sim}{t}'^T \cdot \underset{\sim}{F}^{NT}$ (108)

or, equivalently,

(AMB)→ $\underset{\sim}{h}' \cdot \underset{\sim}{t}^N \cdot \underset{\sim}{\alpha}^N + (\underset{\sim}{h}^N + \underset{\sim}{I}) \cdot (\underset{\sim}{t}' \cdot \underset{\sim}{\alpha}^N + \underset{\sim}{t}^N \cdot \underset{\sim}{\alpha}')$ (109)

= symmetric

(Compatibility)→ $\underset{\sim}{e}' = (\underline{\nabla}^o \underline{\dot{u}})^T$ (110)

or, equivalently, $\underset{\sim}{e}' \equiv \underset{\sim}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) + \underset{\sim}{\alpha}^N \cdot \underset{\sim}{h}' = (\nabla^o \underline{\dot{u}})^T$ (111)

(TBC)→ $\underline{n} \cdot \underset{\sim}{t}' \equiv \underline{t}' = \underline{\bar{t}}'$ at S_{σ_o} (112)

(DBC)→ $\underline{\dot{u}} = \underline{\dot{\bar{u}}}$ at S_{u_o} (113)

3.2.1: General Variational Principles in TL Rate Form:

3.2.1.1: In terms of $\underset{\sim}{s}'$; $\underset{\sim}{E}'$; and $\dot{\underline{u}}$

Using procedures analogous to those leading to Eq. (42) of the UL case, a general rate variational principle governing Eqs. (102 to 106, and 98b) can be stated through the condition of stationarity of the functional*

$$\pi^2_{HW}(\underset{\sim}{s}';\underset{\sim}{E}'; \text{ and } \dot{\underline{u}}) = \int_{V_o} \Big\{W'(\underset{\sim}{E}') - \rho^o\underline{B}' \cdot \dot{\underline{u}} + \frac{1}{2}\underset{\sim}{s}^N:(\underset{\sim}{e}'^T \cdot \underset{\sim}{e}') - \underset{\sim}{s}':[\underset{\sim}{E}' - 1/2(\underset{\sim}{e}' + \underset{\sim}{e}'^T + \underset{\sim}{e}'^T \cdot \underset{\sim}{e}^N + \underset{\sim}{e}^{NT} \cdot \underset{\sim}{e}')]\Big\}dV - \int_{S_{\sigma_o}} \overline{\underline{t}}' \cdot \dot{\underline{u}}\, ds - \int_{S_{u_o}} \underline{t}' \cdot (\dot{\underline{u}} - \dot{\overline{\underline{u}}})\, ds \tag{114}$$

where $\underset{\sim}{e}' = (\underline{\nabla}^o\dot{\underline{u}})^T$; W' is as defined in Eq. (98a), and $\underline{t}'$ is as defined in Eq. (105). Once again, a functional π^1_{HW} to check the true satisfaction of the fully nonlinear field equations in C_N can be derived, but is omitted here [see MURAKAWA (1978),for instance, for further details]. If one eliminates $\underset{\sim}{E}'$ and $\underset{\sim}{s}'$ from Eq. (114), by a priori satisfying Eqs. (98b, 104, and 106), one obtains a potential energy rate principle with the associated functional:

$$\pi^2_p(\dot{\underline{u}}) = \int_{V_o} [W'(\dot{\underline{u}}) - \rho^o\underline{B}' \cdot \dot{\underline{u}} + 1/2\, \underset{\sim}{s}^N: (\underset{\sim}{e}'^T \cdot \underset{\sim}{e}')]dV - \int_{S_{\sigma_o}} \overline{\underline{t}}' \cdot \dot{\underline{u}}\, ds. \tag{115}$$

where $\underset{\sim}{e}' \equiv (\underline{\nabla}^o\dot{\underline{u}})^T$. Likewise, by inverting Eq. 98b, one can achieve a contact transformation,

$$W' - \underset{\sim}{s}':\underset{\sim}{E}' = -S'^*(\underset{\sim}{s}') \tag{116}$$

*This functional can be modified, in a manner analogous to that leading to Eqs. (82 and 85) respectively, to derive rate principles which can be used to treat cases of near or precise incompressibility.

such that $\underset{\sim}{E}' = \partial S'^{*}/\partial \underset{\sim}{s}'$ (117)

Using Eq. (116), one may eliminate $\underset{\sim}{E}'$ as a variable from Eq. (114) and obtain a functional, say $\pi^2_{HR}(\dot{\underline{u}}; \underset{\sim}{s}')$ corresponding to a HELLINGER-REISSNER type rate principle.

In general, a complementary energy rate principle may be derived from Eq. (114) by eliminating $\underset{\sim}{E}'$ from Eq. (114) using Eq. (116); and by satisfying both the equations of LMB and AMB, Eqs. (102) and (103) respectively, a priori. When this is done, one can formally obtain a complementary energy rate functional,

$$\pi^2_c(\dot{\underline{u}}, \underset{\sim}{s}') = \int_{V_o} \left\{ S'^{*}(\underset{\sim}{s}') + \frac{1}{2} \underset{\sim}{s}^N : [(\underline{\nabla}^o \dot{\underline{u}}) \, . \, (\nabla^o \dot{\underline{u}})^T] \right\} dV$$

$$- \int_{S_{u_o}} \underset{\sim}{t}' \, . \, \dot{\underline{\bar{u}}} \, ds \qquad (118)$$

Thus, as in the UL rate formulation, even in the TL formulation, both $\underset{\sim}{s}'$ and $\dot{\underline{u}}$ appear as variables in the complementary principle; however there is a significant difference between the two cases from the point of view of application. In the TL rate formulation, the AMB condition, Eq. (103), is quite simple to be satisfied, provided the chosen $\underset{\sim}{s}'$ is symmetric. However, in LMB condition, Eq. (102), both the stress rate $\underset{\sim}{s}'$, and the displacement gradient rate $\underset{\sim}{e}'$ are involved; moreover, there is a strong coupling between $\underset{\sim}{s}'$ and the currently known functions, $\underset{\sim}{F}^N(\underline{x})$. Thus the admissible stress field $\underset{\sim}{s}'$, to be used in a complementary energy rate principle, if one were contemplated based on $\underset{\sim}{s}'$, must represent a solution to the set of partial differential equations, Eq. (102), with variable coefficients. While this may mathematically not be impossible, it defeats the very purpose of a variational principle forming the basis of a simple numerical method such as a finite element method. Thus the rate complementary energy principle based on $\underset{\sim}{s}'$, for finite strain plasticity analysis, does not appear to be practically useful.

3.2.1.2: Based on $\underset{\sim}{t}'$; $\underset{\sim}{e}'$; and $\dot{\underline{u}}$:

Likewise, a general variational principle governing Eqs. (107, 108, 110, 112, 113 and 100b) can be shown to be governed by the stationarity condition of the functional:

$$\pi^2_{HW}(\underset{\sim}{t}'; \underset{\sim}{e}'; \dot{\underline{u}}) = \int_{V_o} \left\{U'(\underset{\sim}{e}') - \rho_o \underline{B}' \cdot \dot{\underline{u}} - \underset{\sim}{t}'^T : [\underset{\sim}{e}' - (\underline{\nabla}^o \dot{\underline{u}})^T]\right\} dV$$

$$- \int_{S_{\sigma_o}} \overline{\underline{t}}' \cdot \dot{\underline{u}}\, ds - \int_{S_{u_o}} \underline{t}' \cdot (\underline{u}' - \overline{\underline{u}}')\, ds \qquad (119)$$

where U' is defined through Eq. (100a); and $\underline{t}'$ is defined in Eq. (112). As in the UL rate case, because of the special structure of U' as given in Eq. (100a), it can be shown that $\underset{\sim}{t}'$, as derived from U' through Eq. (100b), identically satisfies the AMB condition, Eq. (108).

If from Eq. (119) one eliminates $\underset{\sim}{e}'$ and $\underset{\sim}{t}'$ as variables, by a priori satisfying Eqs. (100b, 110, and 113), one can derive a rate form of a potential energy functional:

$$\pi^2_p(\dot{\underline{u}}) = \int_{V_o} \left\{U'(\dot{\underline{u}}) - \rho^o \underline{B}' \cdot \dot{\underline{u}}\right\} dv - \int_{S_{\sigma_o}} \overline{\underline{t}}' \cdot \dot{\underline{u}}\, ds \qquad (120)$$

the stationarity of which functional leads to Eqs. (107, 108, and 112) as its Euler equations and natural b.c. However, the above variational principle can be seen to identical to that in Eq. (115), because of Eq. (100a).

By inverting Eq. (100b), one may, under certain conditions analogous to those discussed in the UL rate case, achieve a contact transformation,

$$U'(\underset{\sim}{e}') - \underset{\sim}{t}'^T : \underset{\sim}{e}' = -T'^*(\underset{\sim}{t}') \tag{121}$$

Such that $\partial T'^* / \partial \underset{\sim}{t}' = \underset{\sim}{e}'^T$ (122)

However, analogous to the situation discussed earlier in connection with the UL rate formulation, the AMB condition, Eq. (108), cannot be verified to be embedded in the structure of $T'^*(\underset{\sim}{t}')$ as obtained from Eq. (121). Thus the HELLINGER-REISSNER type principle in terms of $\underset{\sim}{t}'$ and $\dot{\underline{u}}$ [derivable by using Eq. (121) in Eq. (119)], or the complementary energy rate principle in terms of $\underset{\sim}{t}'$ alone [derivable by satisfying conditions of Eqs. (121, 107, and 112) a priori in Eq. (119)] cease to be rational principles; since, the AMB condition for $\underset{\sim}{t}'$ is neither embedded in the structure of T'^* nor does it follow as an Euler equation from these principles.

3.2.1.3: In terms of $\underset{\sim}{\alpha}'$; $\underset{\sim}{h}'$; $\underline{\dot{u}}'$; and $\underset{\sim}{r}'(\underset{\sim}{t}'; \underset{\sim}{\alpha}')$

Once again, to avoid the above difficulties in formulating consistent complementary energy and Hellinger-Reissner type rate variational principles, we transform the general variational principle associated with Eq. (119) into one involving $\underset{\sim}{r}'$; $\underset{\sim}{\alpha}'$; $\underset{\sim}{h}'$; and $\dot{\underline{u}}$ as variables. To this end we first note from Eqs. (100a) and (101a),

$$U' = Q' - 1/2\, \underset{\sim}{s}^N : (\underset{\sim}{h}' \cdot \underset{\sim}{h}') + 1/2\, \underset{\sim}{s}^N : (\underset{\sim}{e}'^T \cdot \underset{\sim}{e}') \tag{123}$$

Upon making use of the relations,

$$\underset{\sim}{e}' = \underset{\sim}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) + \underset{\sim}{\alpha}^N \cdot \underset{\sim}{h}' \tag{124}$$

and the orthogonality condition and its rate form;

$$\underset{\sim}{\alpha}^{NT} \cdot \underset{\sim}{\alpha}^N = I;\ \underset{\sim}{\alpha}'^T \cdot \underset{\sim}{\alpha}^N = -\underset{\sim}{\alpha}^{NT} \cdot \underset{\sim}{\alpha}' \tag{125}$$

and the relation $\underset{\sim}{t}^N = \underset{\sim}{s}^N \cdot \underset{\sim}{F}^{NT}$, one can, through relatively straightforward algebra, reduce Eq. (123) to

$$U' = Q' - \underset{\sim}{t}^{NT} : (\underset{\sim}{\alpha}' \cdot \underset{\sim}{h}') - 1/2\, \underset{\sim}{t}^{NT} : [\underset{\sim}{\alpha}' \cdot \underset{\sim}{\alpha}^{NT} \cdot \underset{\sim}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N)] \tag{126}$$

Upon using Eq. (126) and (124), Eq. (119) can be rewritten as:

$$\begin{aligned} \pi^2_{HW}(\dot{\underline{u}}; \underset{\sim}{h}'; \underset{\sim}{\alpha}'; \underset{\sim}{t}') = \int_{V_o} \Big\{ & Q'(\underset{\sim}{h}') - \rho^o \underline{B}' \cdot \dot{\underline{u}} \\ & + \underset{\sim}{t}'^T : [(\underline{\nabla}^o \dot{\underline{u}})^T - \underset{\sim}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) - \underset{\sim}{\alpha}^N \cdot \underset{\sim}{h}'] \\ & - \underset{\sim}{t}^{NT} : (\underset{\sim}{\alpha}' \cdot \underset{\sim}{h}') - 1/2\, \underset{\sim}{t}^{NT} : [\underset{\sim}{\alpha}' \cdot \underset{\sim}{\alpha}^{NT} \cdot \underset{\sim}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N)] \Big\} dV \\ & - \int_{S_{\sigma_o}} \underline{\bar{t}}' \cdot \dot{\underline{u}}\, ds - \int_{S_{u_o}} \underline{t}' \cdot (\dot{\underline{u}} - \dot{\bar{\underline{u}}})\, ds \end{aligned} \tag{127}$$

Where $Q'(\underset{\sim}{h}')$ is the potential for $\underset{\sim}{r}'$ as defined in Eq. (101a) and $\underset{\sim}{r}'$ is to be related to $\underset{\sim}{t}'$ and $\underset{\sim}{\alpha}'$ through Eq. (94). Noting that the variations $\delta\underset{\sim}{\alpha}'$ are required to satisfy the property of Eq. (125b), viz., that $\underset{\sim}{\alpha}^{NT} \cdot \underset{\sim}{\alpha}'$ is skew symmetric, it can be easily shown that the stationarity condition of the above functional yields the Euler equations and natural b.c: (i) the LMB condition, Eq. (107); (ii) the AMB condition, Eq. (109); (iii) the compatibility condition, Eq. (111); (iv) the rate constitutive law, Eq. (101b); (v) the TBC, Eq. (112), and (vi) the DBC Eq. (113).

By inverting (even if numerically) the relation of Eq. (101b), one can obtain a contact transformation,

$$Q' - \underset{\sim}{r}' : \underset{\sim}{h}' = -R'^*(\underset{\sim}{r}')$$

or

$$Q' - 1/2\, [\underset{\sim}{t}' \cdot \underset{\sim}{\alpha}^N + \underset{\sim}{t}^N \cdot \underset{\sim}{\alpha}' + \underset{\sim}{\alpha}^{NT} \cdot \underset{\sim}{t}'^T + \underset{\sim}{\alpha}'^T \cdot \underset{\sim}{t}^{NT}] : \underset{\sim}{h}' = -R'^*(\underset{\sim}{r}') \tag{128}$$

such that $\partial R'^{*}/\partial \underset{\sim}{r}' = \underset{\sim}{h}'$ (129)

By using Eq. (128), one can eliminate $\underset{\sim}{h}'$ as a variable from Eq. (127) and obtain a functional $\pi^2_{HR}(\underline{u}'; \underset{\sim}{\alpha}'; \underset{\sim}{t}')$ corresponding to a HELLINGER-REISSNER type variational principle.

Finally, by requiring $\underset{\sim}{t}'$ to satisfy only the linear LMB condition, Eq. (107), and the TBC, Eq. (112), one can eliminate $\underline{u}'$ as a variable from π^2_{HR} and obtain a TL rate complementary energy functional,

$$\pi^2_c(\underset{\sim}{t}';\underset{\sim}{\alpha}') = \int_{V_o} \Big\{-R'^{*}(\underset{\sim}{r}') - \underset{\sim}{t}'^{T}:[\underset{\sim}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N)]$$

$$- \frac{1}{2}\underset{\sim}{t}^{NT}:[\underset{\sim}{\alpha}' \cdot \underset{\sim}{\alpha}^{NT} \cdot \underset{\sim}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N)]\Big\}dV$$

$$+ \int_{S_{u_o}} \underline{t}' \cdot \dot{\underline{u}}\, ds \tag{130}$$

Wherein it is implied that $\underset{\sim}{r}'$ is related to $\underset{\sim}{t}'$ and $\underset{\sim}{\alpha}'$ through Eq. (94). Noting that the variations $\delta\underset{\sim}{t}'$ are now subject to the constraints that $\underline{\nabla}^o \cdot \delta\underset{\sim}{t}' = 0$ in V_o and $\underline{n} \cdot \delta\underset{\sim}{t}' = 0$ at S_{σ_o}; and that the variation $\delta\underset{\sim}{\alpha}'$ are subject to the constraint that $\underset{\sim}{\alpha}^{NT} \cdot \delta\underset{\sim}{\alpha}'$ is skew-symmetric, it can be shown easily that the condition of vanishing of the first variation of the above functional leads to: (i) compatibility condition, $(\nabla^o\dot{\underset{\sim}{u}})^T = \underset{\sim}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N) + \underset{\sim}{\alpha}^N \cdot \underset{\sim}{h}'$; (ii) the AMB condition, Eq. (109); (iii) the DBC, Eq. (113).

Once again, in as much as the AMB condition follows unambiguously as in Euler equation; and the admissible $\underset{\sim}{t}'$ is required to satisfy only the uncoupled linear LMB condition, Eq. (107) [which can be satisfied easily through first-order stress functions, $\underset{\sim}{\psi}$, as $\underset{\sim}{t}' = \underline{\nabla}^o \times \underset{\sim}{\psi} + \underset{\sim}{t}'^{P}$; and $\underset{\sim}{t}'^{P}$ is any particular solution such that $\underline{\nabla}^o \cdot \underset{\sim}{t}'^{P} = -\rho^o\underline{B}'$], and the TBC, Eq. (12), the

TL rate complementary energy principle of Eq. (130), is the most consistent and useful principle for purposes of engineering application.

As in Eq. (75) for the UL rate case, in the application of Eq. (130) also to a finite element assemblage, the interelement traction reciprocity condition, $(\underline{n} \cdot \underset{\sim}{t}')^+ + (\underline{n} \cdot \underset{\sim}{t}')^-$ at ρ_{mo} can be relaxed a priori, and introduced as a constraint condition in a modified complementary energy principle through Lagrange Multipliers $\dot{\tilde{\underline{u}}}_p$ at ρ_{mo}. This modified principle is stated through the functional:

$$\pi^2_{mc}(\underset{\sim}{t}'; \underset{\approx}{\alpha}'; \dot{\tilde{\underline{u}}}_p) = \sum_m \int_{V_{mo}} \left\{ -R'^*(\underset{\sim}{r}') - \underset{\sim}{t}'^T : [\underset{\approx}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N)] \right.$$

$$\left. - \frac{1}{2} \underset{\sim}{t}^{NT} : [\underset{\approx}{\alpha}' \, \underset{\sim}{\alpha}^{NT} \cdot \underset{\sim}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^N)] \right\} dV$$

$$+ \sum_m \int_{S_{u_{mo}}} \underline{t}' \cdot \dot{\underline{u}} \, ds + \sum_m \int_{\rho_{mo}} (\underline{n} \cdot \underset{\sim}{t}') \cdot \dot{\tilde{\underline{u}}}_p \, d\rho \tag{131}$$

As in the UL rate case [see discussion following Eq. (75)] the functional in Eq. (31) can be used to develop a finite element "stiffness matrix" method and Eq. (130) can be used to develop finite element "flexibility matrix" approach. The author's work in progress in this regard, along with analysis of certain metal forming problems is the subject of a forthcoming paper.

Finally it should be remarked that eventhough the development of Eq. (130), as a basis of a TL rate complementary energy principle for finite strain elasto-plasticity was based on independent considerations, the result is analogous to the principle derived by FRAEIJS DE VEUBEKE (1972). However, FREIJS DE VEUBEKE'S (1972) principle governs the total (as opposed to rates) deformations of a compressible elastic solid. It is also noted that a TL rate principle equivalent to that of FRAEIJS DE VEUBEKE (1972) was developed and used, in the context of an assumed stress-rate finite element method, to solve certain finite strain problems for nonlinear elastic, compressible as well as incompressible solids by MURAKAWA and ATLURI (1978a,b).

Finally we remark that the TL rate complementary energy principle for elastic-plastic solids given presently in Eq. (130) differs slightly from that for nonlinear elastic solids given eariler by MURAKAWA and ATLURI (1978a), this difference is in the third term in the volume integrand on the right hand side of Eq. (130); instead of the term appearing in Eq. (130), the term $-\underset{\sim}{t}^{NT}:[\underset{\sim}{\alpha}' \cdot (\underset{\sim}{I} + \underset{\sim}{h}^{N})]$ appears in the paper by MURAKAWA and ATLURI (1978a). The effect of this is: whereas the exact rate condition of AMB, Eq. (109) becomes an Euler equation of the principle in Eq. (130), the AMB condition which follows as an Euler Equation for the principle given in MURAKAWA and ATLURI (1978a) is that $(\underset{\sim}{I} + \underset{\sim}{h}^{N})' \cdot \underset{\sim}{t}' \cdot \underset{\sim}{\alpha}^{N} + \underset{\sim}{h}' \cdot \underset{\sim}{t}^{N} \cdot \underset{\sim}{\alpha}^{N}$ is symmetric [which condition represents only an approximation to the exact condition, Eq. (109)]. However, the iterative correction procedures to check the satisfaction of the fully nonlinear AMB condition of Eq. (11a) at the end of each increment were employed by MURAKAWA and ATLURI (1978a) to correct the above approximation in the rate AMB condition. In this sense, the principle currently stated through Eq. (130) [which is equally applicable to nonlinear elastic solids, when the potential $R'^{*}(\underset{\sim}{r}')$ is appropriately defined] is the most consistent TL rate complementary energy principle for finite strain analysis of elastic as well as elastic-plastic solids.

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General Variational theorems, for the rate problem of classical elasto-plasticity at finite strains, in both Updated Lagrangean and Total Lagrangean rate forms, and in terms of alternate measures of stress-rate and conjugate strain-rates, are critically studied from the point of view of their application. Attention is primarily focused on the derivation of consistent complementary energy rate principles, which could form the basis of consistent and rational assumed stress type finite element methods; and two → next page

such principles, in both UL and TL forms, are newly stated. Systematic procedures to exploit these new principles in the context of a finite element method are also discussed. Also, certain general modified variational theorems, to enable an accurate numerical treatment of near incompressible behavior at large plastic strains, are discussed.